

Supercharacters of unipotent and solvable groups

A.N.Panov *

Abstract

The notion of the supercharacter theory was introduced by P.Diaconis and I.M.Isaacs in 2008. In this paper we review the main statements of the general theory, we observe the construction of supercharacter theory for algebra groups and the theory of basic characters for the unitriangular groups over the finite field. Basing on the previous papers of the author, we construct the supercharacter theory for the finite groups of triangular type. We characterize the structure of Hopf algebra of supercharacters for the triangular group over the finite field.

1 Introduction

Traditionally, the main problem of the representation theory of finite groups is as follows: given a finite group, to classify all its irreducible representations (characters). It turns out that for some finite groups this problem is extremely difficult, its solution is unknown and there are no ideas how to solve it in future. The main example of these groups is the unitriangular group $UT_n = UT(n, \mathbb{F}_q)$ over the finite field of q elements. This group is an unipotent matrix group, and it is known the orbit method of A.A.Kirillov is valid for these groups [22, 23, 21]. According to this method, there exists one to one correspondence between the irreducible representations and the orbits of coadjoint representation. This correspondence plays an important role in the representation theory because it enables to produce a solution of the decomposition problem for the representations obtained by restriction and induction from the irreducible ones. Moreover, the character of irreducible representation can be directly presented as a sum taken over elements of the coadjoint orbit (the formula of A.A.Kirillov). However, it is not easier to classify the coadjoint orbits than to classify the irreducible representations. It is reasonable to say that the orbit method provides the equivalence of categories, but not the classification of irreducible representations.

In the series of papers [2, 3, 4, 5], in 1995-2003, C.André investigated the theory of basic characters of the unitriangular group UT_n . Although the basic characters are not irreducible in general, the system of basic characters has many common features with the system of irreducible characters: these characters are pairwise disjoint, they are constant on some system of subsets

*The research is supported by the grant RSF-DFG 16-41-1013

(called the system of basic subvarieties in UT_n); the number of basic characters is equal to the number of basic subvarieties, this enables to construct the quadratic table of values (the basic character table). Each basic subvariety in UT_n is a union of the classes of conjugate elements. One can correspond the system of basic characters to some partition of the dual space \mathfrak{ut}_n^* into the subsets, which C.André called the basic subvarieties in \mathfrak{ut}_n^* (the analog of the orbit method). In contrast to irreducible representations, the basic characters admit the exact presentation in terms of the basic subsets in the system of positive roots of series A_n . Observe that C.André defined a basic subvariety in \mathfrak{ut}_n^* as a sum of elementary coadjoint orbits. Apparently, this approach is universal and is valid for the other series of simple Lie algebras, but it is difficult for calculating. In his thesis, Ning Yan [37] observed that for the series A_n the basic subvarieties in \mathfrak{ut}_n^* coincide with the orbits of double left-right action of the group UT_n on \mathfrak{ut}_n^* .

In the paper [16], in 2008, P.Diaconis and I.M.Isaaks presented the notion of a supercharacter theory of an arbitrary finite group. They set the properties of the basic characters of the unitriangular group as axioms of a supercharacter theory. Roughly speaking, to construct a supercharacter theory for a given finite group is to construct the system of disjoint characters (supercharacters) χ_1, \dots, χ_m and the partition of the group into subsets (superclasses) K_1, \dots, K_m such that the supercharacters are constant on superclasses. The main goal is to construct the supercharacter theory which produces the best approximation of the theory of irreducible representations. In the paper [16], P.Diaconis and I.M.Isaaks constructed the supercharacter theory for algebra groups, its partial case is the theory of basic characters for UT_n .

Many papers were devoted to different supercharacter theories. Observe a few of them: supercharacters for abelian groups and their application in the number theory [14, 17], the superinduction for the algebra groups [34, 35, 25], the supercharacter theory for Sylow subgroups in orthogonal and symplectic groups over a finite field [10], application in the random walk problem on groups [11], the supercharacter theory for semidirect products [19], characterization of the Hopf algebra of supercharacters for the unitriangular group [1, 13]. One can found the bibliography in the paper [1].

The most of papers are devoted either to general questions of supercharacter theories, or to constructions of supercharacter theories for the groups closely related to UT_n . One of the main open problems is to enlarge the list, to construct an appropriate supercharacter theory for such groups as the parabolic subgroups in finite Chevalley groups. Observe that the Mackey

method of classification of irreducible representations of semidirect products apparently failed for supercharacters. This is the main obstacle here. The general approach of the paper [19] provides rather rough supercharacter theory, which can be improved in examples.

The most of this paper is a survey. Based on the paper [16] of P.Diaconis and I.M.Isaaks we observe the general questions of supercharacter theories and construct the supercharacter theory of algebra groups (see section 2 and 3). We also study the theory of basic characters of C.André and characterize the Hopf algebra of supercharacters of the unitriangular group following the paper [1] (see sections 3 and 4). In section 5, we expound the supercharacter theory for the finite groups of triangular type following the author papers [28, 29, 30]. In section 6, we characterize the Hopf algebra of supercharacters of the triangular group in terms of the Hopf algebra of partially symmetric functions in noncommuting variables.

2 Supercharacters and superclasses

Let G be a finite group, $1 \in G$ be its identity element, $\text{Irr}(G)$ be the set of all irreducible characters (representations) of the group G . Given two partitions

$$\text{Irr}(G) = X_1 \cup \cdots \cup X_m, \quad X_i \cap X_j = \emptyset, \quad (1)$$

$$G = K_1 \cup \cdots \cup K_m, \quad K_i \cap K_j = \emptyset. \quad (2)$$

Observe that the partitions have equal number of components. We correspond X_i to the character of the group G by the formula

$$\sigma_i = \sum_{\psi \in X_i} \psi(1)\psi. \quad (3)$$

Definition 2.1. Two partitions $\mathcal{X} = \{X_i\}$ and $\mathcal{K} = \{K_j\}$ is said to define a supercharacter theory of the group G if each character σ_i is constant on each K_j . In this case, $\{\sigma_i\}$ are called *supercharacters*, and $\{K_j\}$ *superclasses*. The table of values $\{\sigma_i(K_j)\}$ is called the supercharacter table.

The next proposition is suitable for a construction of examples of supercharacter theories.

Proposition 2.2 [16, 2.1]. Let we have the system of disjoint characters $\mathfrak{Ch} = \{\chi_1, \dots, \chi_m\}$ and the partition $\mathcal{K} = \{K_1, \dots, K_m\}$ of the group G . Suppose that each character χ_i is constant on each K_j . Let X_i denote the *support of character* χ_i (i.e. the set of all irreducible components of χ_i). Then the following conditions are equivalent:

- 1) $\{1\} \in \mathcal{K}$,

2) the system of subsets $\mathcal{X} = \{X_i\}$ is a partition of $\text{Irr}(G)$; the partitions \mathcal{X} and \mathcal{K} define a supercharacter theory of the group G . Moreover, each χ_i equals to σ_i up to a constant multiplier. Simplifying language, we refer to χ_i as a *supercharacter*.

Remark. Recall that the characters are disjoint (i.e. their supports do not intersect) whenever they are orthogonal.

Proof. Assume that the condition 1) is fulfilled. Any system of disjoint characters is linearly independent. The number of characters from \mathfrak{Ch} equals to the number of sets from \mathcal{K} . Hence, \mathfrak{Ch} is a basis in the space of all complex valued function on G constant on subsets from \mathcal{K} . The regular character $\rho(g)$ equals to $|G|$ if $g = 1$ and equals to 0 if $g \neq 1$. Since $\{1\} \in \mathcal{K}$, the character $\rho(g)$ is constant on the subsets from \mathcal{K} . We obtain

$$\rho = \sum_{i=1}^m a_i \chi_i, \quad a_i \in \mathbb{C}.$$

On the other hand, any irreducible character occurs in decomposition of ρ with multiplicity equals to its degree. Therefore, any irreducible character occurs in decomposition of exactly one χ_i and $a_i \chi_i = \sigma_i$, $a_i \in \mathbb{Q}^*$. This proves 2).

Assume that the condition 2) is fulfilled. The identity element of the group belongs to exactly one of the subsets of \mathcal{K} , say K_1 . Since the characters χ_i are constant on superclasses, $\chi_i(g) = \chi_i(1)$ for each $g \in K_1$ and $1 \leq i \leq m$. The condition 2) implies $\rho(g) = \rho(1)$ and then $g = 1$ and $K_1 = \{1\}$. \square

Corollary 2.3. The system of supercharacters $\mathfrak{Ch} = \{\chi_i\}$ is determined up to constants multipliers by the partition $\mathcal{K} = \{X_i\}$.

Present some examples of supercharacter theories.

Example 2.4. The system of irreducible characters $\{\chi_i\}$ and the system of classes of conjugate elements $\{K_i\}$.

Example 2.5. Two supercharacters $\chi_1 = 1_G$, $\chi_2 = \rho - 1_G$ (here ρ is the character of regular representation) and two superclasses $K_1 = \{1\}$, $K_2 = G \setminus \{1\}$. Here is the table of supercharacters

	χ_1	χ_2
K_1	1	$ G - 1$
K_2	1	-1

Example 2.6. $G = C_4$ is the cyclic group of 4th order. Below we present the table of irreducible characters and the one of supercharacter tables.

	χ_0	χ_1	χ_2	χ_3
1	1	1	1	1
g	1	i	-1	-i
g^2	1	-1	1	-1
g^3	1	-i	-1	i

	χ_0	$\chi_1 + \chi_3$	χ_2
1	1	2	1
$\{g, g^3\}$	1	0	-1
g^2	1	-2	1

Example 2.7. If the group Γ acts on G and $\text{Irr}(G)$, and these actions agree $\chi^a(g^a) = \chi(g)$, then two partitions

$$\chi^\Gamma = \sum_{a \in \Gamma} \chi^a, \quad K^\Gamma = \bigcup_{a \in \Gamma} K^a,$$

where χ (respectively, K) run through the set of representatives of Γ -orbits in $\text{Irr}(G)$ (respectively, the classes of conjugate elements in G), give rise to a supercharacter theory of the group G . The partial case is presented in the previous example: the group $\Gamma = \mathbb{Z}_2$; its generator acts on G and $\text{Irr}(G)$ by raising to the power three.

For the groups of small order, it is possible to enumerate all supercharacter theories. It was proved in the paper [15] that there exist only three groups with exactly two supercharacter theories: the cyclic group Z_3 , the symmetric group S_3 and the simple group $\text{Sp}(6, 2)$. For different supercharacter theories see the paper [12].

Turn to formulating some general statements on supercharacters and superclasses.

Lemma 2.8. If $\{X_1, \dots, X_m\}$ is a partition of $\text{Irr}(G)$, and $\{K_1, \dots, K_t\}$ is a partition of G , and σ_i , $1 \leq i \leq m$, is constant on each K_j , $1 \leq j \leq t$, then $m \leq t$.

Proof. The system of characters $\{\sigma_i\}$ is linearly independent and is contained in the t -dimensional space of all complex valued functions on G constant on $\{K_j\}$. \square

For any subset $K \subseteq G$, we denote

$$\hat{K} = \sum_{g \in K} g.$$

The center $Z(\mathbb{C}G)$ of the group algebra $\mathbb{C}G$ is a direct sum

$$Z(\mathbb{C}G) = \bigoplus_{\psi \in \text{Irr}(G)} \mathbb{C}e_\psi,$$

where $\{e_\psi\}$ is a system of primitive idempotents. The primitive idempotent e_ψ , associated with the irreducible character ψ , can be calculated by the formula

$$e_\psi = \frac{\psi(1)}{|G|} \sum_{g \in G} \overline{\psi(g)} g.$$

We correspond $X \subseteq \text{Irr}(G)$ to the central idempotent

$$f_X = \sum_{\psi \in X} e_\psi. \quad (4)$$

Proposition 2.9 [16]. Let the partitions \mathcal{X} and \mathcal{K} define a supercharacter theory, and $f_i = f_{X_i}$. Then the system $\{\widehat{K_j} : 1 \leq j \leq m\}$ forms a basis in the algebra

$$A(\mathcal{X}, \mathcal{K}) = \text{span}\{f_i : 1 \leq i \leq m\}.$$

Proof. Direct calculations lead to the equality

$$\begin{aligned} f_i &= \sum_{\psi \in X_i} e_\psi = \sum_{\psi \in X_i} \frac{\psi(1)}{|G|} \sum_{g \in G} \overline{\psi(g)} g \\ &= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{\psi \in X_i} \psi(1) \overline{\psi(g)} \right) g = \frac{1}{|G|} \sum_{j=1}^m \overline{\sigma_i(K_j)} \widehat{K_j}. \end{aligned} \quad (5)$$

Since the systems $\{f_i\}$ and $\{\widehat{K_j}\}$ are of equal cardinality and linearly independent, they generate the common subspace. \square

Corollary 2.10. If the partitions \mathcal{X} and \mathcal{K} define a supercharacter theory, then each K_j is invariant with respect to conjugation (i.e. it can be decomposed into union of classes of conjugate elements).

Proof. The Proposition 2.9 implies the elements $\{\widehat{K_j}\}$ are contained in the center of the group algebra $\mathbb{C}G$. \square

Proposition 2.11 [16]. In any supercharacter theory the partitions \mathcal{X} and \mathcal{K} uniquely determine each other.

Proof. 1) The partition (1) of the set $\text{Irr}(G)$ defines an equivalence relation on the group G such that $x \sim y$ if $\sigma_i(x) = \sigma_i(y)$ for any $1 \leq i \leq m$. This relation produces the partition \mathcal{K}^0 of group G . Recall that $\{\sigma_i\}$ is a basis in the space of complex valued functions constant on the partition \mathcal{K} . The partition \mathcal{K} is sharper than \mathcal{K}^0 . Therefore $m = |\mathcal{K}| \geq |\mathcal{K}^0|$. On the other hand, according to Lemma 2.8 $|\mathcal{K}^0| \geq |\mathcal{X}| = m$. Then $|\mathcal{K}^0| = m$ and $\mathcal{K}^0 = \mathcal{K}$.

2) The partition (2) of the group G defines the subalgebra $\text{span}\{\widehat{K_j}\} = \text{span}\{f_i\}$. Its basis of orthogonal idempotents is uniquely determined. Therefore, the partition \mathcal{X} is uniquely defined by \mathcal{K} . \square

Proposition 2.12 [16]. The principal character 1_G is a supercharacter (recall that the character is principal if it is identically equal to one).

Proof. Assume the opposite. Let $1_G \in X_1$ and $X_1 \neq \{1_G\}$. Then the partition \mathcal{X}^0 that differs from \mathcal{X} by decomposition

$$X_1 = \{1_G\} \cup (X_1 \setminus \{1_G\}),$$

forms the pair $(\mathcal{X}^0, \mathcal{K})$ obeying conditions of Proposition 2.2. However, $|\mathcal{X}^0| > |\mathcal{K}|$. This contradicts the conclusion of Lemma 2.8. \square

Proposition 2.13 [16]. Let $\tau \in \text{Aut}(\mathbb{C})$. Then, for any supercharacter theory $(\mathcal{X}, \mathcal{K})$, the automorphism τ acts as a permutation of the partition \mathcal{X} .

Proof. The system $\mathcal{X}^\tau = \{X_i^\tau\}$ is a partition of $\text{Irr}(G)$. The pairs $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{X}^\tau, \mathcal{K})$ defines two supercharacter theories with common \mathcal{K} . By Proposition 2.11, $\mathcal{X} = \mathcal{X}^\tau$. \square

Proposition 2.14 [16]. Let $(r, |G|) = 1$. Then the bijection $g \rightarrow g^r$ permutes superclasses.

Proof. The bijection $g \rightarrow g^r$ transform the partition \mathcal{K} into new partition $\mathcal{K}^r = \{K_i^r\}$. Consider the primitive root of unity ϵ of degree $|G|$ and the automorphism τ of the field \mathbb{C} such that $\tau(\epsilon) = \epsilon^r$. The previous proposition implies that τ acts on the partition \mathcal{X} by permutation and $\sigma_i(K_j^r) = \sigma_i^\tau(K_j)$. The pairs $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{X}, \mathcal{K}^r)$ define two supercharacter theories with common \mathcal{X} . By Proposition 2.11, $\mathcal{K} = \mathcal{K}^r$. \square

Corollary 2.15. The partition \mathcal{K} coincides with the partition $\mathcal{K}^{-1} = \{K_j^{-1}\}$.

Definition 2.16. Let A be a subalgebra in $\mathbb{C}G$. The subalgebra A is called a Schur subalgebra if there exists the partition $\mathcal{K} = \{K_i\}$ of the group G such that

- 1) $\widehat{\mathcal{K}} = \{\widehat{K}_i\}$ is a basis in A ,
- 2) $\{1\} \in \mathcal{K}$,
- 3) $\mathcal{K}^{-1} = \mathcal{K}$.

Theorem 2.17 [19]. The map $(\mathcal{X}, \mathcal{K}) \rightarrow A(\mathcal{X}, \mathcal{K})$ establishes one to one correspondence between the set of supercharacters and the set of Schur subalgebras lying in the center $Z(\mathbb{C}G)$ of the group algebra.

Proof. From Proposition 2.9 and Corollary 2.15 we conclude that $A(\mathcal{X}, \mathcal{K})$ is a Schur subalgebra lying in the center $Z(\mathbb{C}G)$. Let us show the opposite statement. Let A Schur subalgebra in $Z(\mathbb{C}G)$. The algebra $Z(\mathbb{C}G)$ is a direct sum of a few copies of the field \mathbb{C} . The subalgebra A contains 1 and is also a direct sum of a few copies of \mathbb{C} . There exists a partition $\mathcal{X} = \{X_i\}$ of the set $\text{Irr}(G)$ such that the system of idempotents $\{f_i\}$ (see (4)) is a basis of A . The bases $\{f_i\}$ and $\{\widehat{K}_i\}$ have equal cardinality. Then $|\mathcal{X}| = |\mathcal{K}|$. By the formulas (5), we obtain $\sigma_i(K_j) = \text{const}$. The partitions $(\mathcal{X}, \mathcal{K})$ define a supercharacter theory for the group G , and $A = A(\mathcal{X}, \mathcal{K})$. \square

Observe that we don't apply the condition 3) of the definition of Schur algebra in the second part of the proof. Therefore, any subalgebra contained in $Z(\mathbb{C}G)$ and satisfying conditions 1) and 2) is a Schur algebra.

The cited below theorem is an analog of the statement for irreducible characters.

Theorem 2.18 [16]. Let χ be a supercharacter, and K be a superclass. Then $\frac{\chi(g)|K|}{\chi(1)}$ is an integer algebraic number.

Let us expound the construction of $*$ -product of supercharacters from the paper [19], which plays an impotent role in constructions of different supercharacter theories. Let G be a finite group, N be a normal subgroup in G , $\pi : G \rightarrow G/N$ be the natural projection.

A supercharacter theory $(\mathcal{X}, \mathcal{K})$ of N is called G -invariant if the action G on N and $\text{Irr}(N)$ preserve the partitions \mathcal{X} and \mathcal{K} . Observe that, in a given supercharacter theory, the partitions G and $\text{Irr}(G)$ are uniquely determine each other (see Proposition 2.11). Therefore, it is sufficient to require invariance of the only one of two partitions (\mathcal{X} or \mathcal{K}).

Let we have the G -invariant supercharacter theory $(\mathcal{X}, \mathcal{K})$ of the group N , where $\text{Irr}(N) = X_1 \cup \dots \cup X_m$ with $X_1 = \{1_N\}$ and $N = K_1 \cup \dots \cup K_m$. Let we have an arbitrary supercharacter theory $(\mathcal{Y}, \mathcal{L})$ of the factor group G/N , where $\text{Irr}(G/N) = Y_1 \cup \dots \cup Y_k$ and $G/N = L_1 \cup \dots \cup L_k$ with $L_1 = \{1\}$. For any subset $X \subset \text{Irr}(N)$, we denote by $\text{Ind}(X, G)$ the sum of induced characters $\text{Ind}(\psi, G)$, where $\psi \in X$. Consider the partitions

$$\text{Irr}(G) = \text{supp}(\text{Ind}(X_2, G)) \cup \dots \cup \text{supp}(\text{Ind}(X_m, G)) \cup Y_1 \cup \dots \cup Y_k. \quad (6)$$

$$G = K_1 \cup \dots \cup K_m \cup \pi^{-1}(L_2) \cup \dots \cup \pi^{-1}(L_k). \quad (7)$$

Theorem 2.19 [19]. The partitions (6) and (7) give rise to a supercharacter theory of the group G .

This supercharacter theory is called a $*$ -product of the supercharacter theories $(\mathcal{X}, \mathcal{K})$ of N and $(\mathcal{Y}, \mathcal{L})$ of the factor group G/N . Observe that this supercharacter theory is rather rough in examples. In some papers, the construction of $*$ -product was modified to obtain a more sharp supercharacter theory.

3 Supercharacters of finite unipotent groups

3.1 Theory of supercharacters for algebra groups

The supercharacter theory for algebra groups was constructed by P.Diaconis and I.M.Isaaks in the paper [16]. By definition, an *algebra group* is a group of the form $G = 1 + J$, where J is an associative finite dimensional nilpotent algebra over the finite field \mathbb{F}_q . The superclass of the element $1 + x$ is defined as $1 + \omega$, where ω is a left-right $G \times G$ -orbit of the element $x \in J$. One can define also the left and right actions of the group G on the dual space J^* by the formulas $\lambda g(x) = \lambda(gx)$ and $g\lambda(x) = \lambda(xg)$. Let $G_{\lambda, \text{right}}$ be the stabilizer of $\lambda \in J^*$ with respect to the right action of G on J^* .

Fix the nontrivial character $t \rightarrow \varepsilon^t$ of the additive group of field \mathbb{F}_q into

the multiplicative group \mathbb{C}^* . The function $\xi : G_{\lambda, \text{right}} \rightarrow \mathbb{C}^*$, defined by

$$\xi_\lambda(g) = \varepsilon^{\lambda(g-1)},$$

is a linear character (one dimensional representation) of the group $G_{\lambda, \text{right}}$. Indeed, if $g_1 = 1 + x_1$ and $g_2 = 1 + x_2$ are two elements in $G_{\lambda, \text{right}}$, then $\lambda(x_1 x_2) = 0$

$$\begin{aligned} \xi_\lambda(g_1 g_2) &= \xi_\lambda(1 + x_1 + x_2 + x_1 x_2) = \varepsilon^{\lambda(x_1 + x_2 + x_1 x_2)} = \\ &= \varepsilon^{\lambda(x_1)} \varepsilon^{\lambda(x_2)} \varepsilon^{\lambda(x_1 x_2)} = \varepsilon^{\lambda(x_1)} \varepsilon^{\lambda(x_2)} = \xi_\lambda(g_1) \xi_\lambda(g_2). \end{aligned}$$

A *supercharacter* of the algebra group G is the induced character

$$\chi_\lambda = \text{Ind}(\xi_\lambda, G_{\lambda, \text{right}}, G). \quad (8)$$

Theorem 3.1 [16]. The systems of supercharacters $\{\chi_\lambda\}$ and superclasses $\{1 + GxG\}$, where λ and x run through the systems of representatives of $G \times G$ -orbits in J^* and J respectively, give rise to a supercharacter theory of the group G .

Thus, the characters of the system $\{\chi_\lambda\}$ are pairwise disjoint. The classification problem of irreducible representation reduces to decomposition of supercharacters into a sum of irreducible components. This problem remains open up today. It could be interesting the following result.

Theorem 3.2 [9]. $(\chi_\lambda, \chi_\lambda) = |G\lambda \cap \lambda G|$.

Proof. One can realize the representation with character χ_λ in the space $V_\lambda = \langle f_\mu(1 + x) = \varepsilon^{\mu(x)} : \mu \in G\lambda \rangle$ by the formula $T_g f(s) = f(sg)$. That is, the representation with character χ_λ is a subrepresentation of the right regular representation of the group G in the space of complex valued functions $\mathbb{C}[G]$. The representation in the space $\mathbb{C}[G]$ is completely reducible; $\mathbb{C}[G]$ is decomposed into a direct sum of V_λ and its complement V'_λ . Any $\phi \in \text{Hom}_G(V_\lambda, V_\lambda)$ extends to the intertwining operator $\tilde{\phi}$ in the space $\mathbb{C}[G]$ that is equal to zero on V'_λ . Then $\tilde{\phi}$ is a linear combination of the left translation operators $L_g f(s) = f(gs)$ on the group G . The operator of left translation by $a \in G$ preserves the subspace V_λ whenever $\lambda a \in G\lambda$. Then $|\text{Hom}_G(V_\lambda, V_\lambda)| = |G\lambda \cap \lambda G|$. \square

Let us recall the A.A.Kirillov formula for the irreducible characters. For each $\lambda \in J^*$ one can construct the irreducible character Ψ_λ of the algebra group $G = 1 + J$ (see [22]); it can be presented in the form:

$$\Psi_\lambda(1 + x) = \frac{1}{\sqrt{|\Omega^*|}} \sum_{\mu \in \Omega^*} \varepsilon^{\mu(x)}, \quad (9)$$

where Ω^* is the coadjoint orbit of $\lambda \in J^*$. Observe that this formula is valid if the characteristic of the field is sufficiently great [22], [21].

The functions defined by (9) are called *Kirillov functions*. The number of Kirillov functions equals to the number of irreducible representations (by [16, Lemma 4.1]), however, this sets do not coincide in general. The simplest example: the group $UT(n, \mathbb{F}_q)$, $n \geq 13$, for $\text{char}(\mathbb{F}_q) = 2$ [20].

In the paper [16], the supercharacter analog of the formula (9) for the algebra groups was presented:

$$\chi_\lambda(1+x) = \frac{1}{n(\lambda)} \sum_{\mu \in G\lambda G} \varepsilon^{\mu(x)}, \quad (10)$$

where χ_λ is the supercharacter associated with $\lambda \in J^*$, and $n(\lambda)$ is the number of right G -orbits in $G\lambda G$. This formula is valid over a field of an arbitrary characteristic.

The constant $n(\lambda)$ is of interest for the other reason. Recall that there is the standard supercharacter σ_λ that is the sum of all irreducible constituents of χ_λ with multipliers equal to their degrees (see Proposition 2.2). The supercharacter χ_λ differs from σ_λ by a constant multiplier. It turns out that this constant is equal to $n(\lambda)$, more precisely $n(\lambda)\chi_\lambda = \sigma_\lambda$ [16].

In the same paper [16], the other version of the formula (10) was presented in the form

$$\chi_\lambda(1+x) = \frac{|\lambda G|}{|GxG|} \sum_{y \in GxG} \varepsilon^{\lambda(y)}. \quad (11)$$

Turn to the questions on restriction and induction in the supercharacter theory. Let $G = 1+J$ be an algebra group. If J' is an arbitrary its subalgebra, then the subgroup $G' = 1+J'$ as called an *algebra subgroup* in G .

It is proved in the paper [16] that the restriction of the supercharacter χ_λ on the algebra subgroup G' is a sum of supercharacters of the subgroup G' with non-negative integer coefficients.

Let ϕ be a superclass function on G' (i.e., it is a function constant on superclasses in G'). Extend ϕ to a function ϕ on G taking it equal to zero outside of G' .

By definition, a *superinduction* for ϕ is a function $\text{SInd } \phi$ on the group G defined by the formula

$$\text{SInd } \phi(1+x) = \frac{1}{|G| \cdot |G'|} \sum_{a,b \in G} \phi(1+axb).$$

Easy to see that $\text{SInd } \phi(1+x)$ is a superclass function on G . The scalar product on G (and G') is defined as usual

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}. \quad (12)$$

The following theorem is a supercharacter analog of the Frobenius theorem.

Theorem 3.3 [16]. Let ϕ be a superclass function on G' , and ψ be a superclass function on G . Then $(\text{SInd } \phi, \psi) = (\phi, \text{Res } \psi)$.

3.2 Theory of basic characters of the unitriangular group

The most impotent example of an algebra group is the unitriangular group $\text{UT}_n = \text{UT}(n, \mathbb{F}_q)$ that consists of all upper triangular matrices the entries from the finite field \mathbb{F}_q and with ones on the diagonal. Its Lie algebra $\mathfrak{ut}_n = \mathfrak{ut}(n, \mathbb{F}_q)$ consists of all strong upper triangular matrices, and, indeed, it is an associative algebra. The group $\text{UT}_n = 1 + \mathfrak{ut}_n$ is an algebra group. Created by C.André the theory of basic characters [2, 3, 4] is a special case of the P.Dianconis and I.M.Isaaks theory of supercharacters for algebra groups [16].

Introduce a few definitions. A *root* is a pair of positive integers (i, j) , where $1 \leq i, j \leq n$ and $i \neq j$. The root (i, j) is *positive* (respectively, negative) if $i < j$ (respectively, if $i > j$). The number i is called a *number of row* of the root $\alpha = (i, j)$ (denote $i = \text{row}(\alpha)$). Respectively, j is a *number of column* (denote $j = \text{col}(\alpha)$). The Lie algebra \mathfrak{ut}_n has the basis of matrix units $\{E_\alpha : \alpha > 0\}$.

The subset of positive roots D is called a *basic subset* (after C.André) if it has at most one root in each row and each column. The other name of D is a *subset of rooks arrangement type*. Suppose that we have a map $\phi : D \rightarrow \mathbb{F}_q^*$. The pair (D, ϕ) is called an *admissible pair* if D is a basic subset.

For each admissible pair (D, ϕ) , we consider the element

$$X_{D, \phi} = \sum_{\alpha \in D} \phi(\alpha) E_\alpha$$

in the Lie algebra \mathfrak{ut}_n . The dual space \mathfrak{ut}_n^* has the basis $\{E_\alpha^* : \alpha > 0\}$ dual to $\{E_\alpha : \alpha > 0\}$. Denote by $\lambda_{D, \phi}$ the element of the dual space of the form

$$\lambda_{D, \phi} = \sum_{\alpha \in D} \phi(\alpha) E_\alpha^*.$$

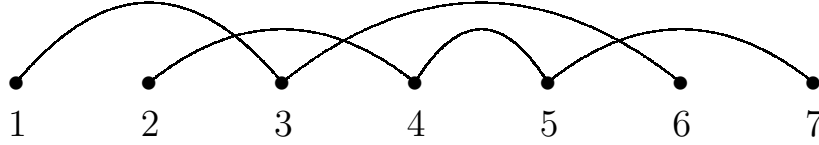
Theorem 3.4 [16, 37]. Let $G = \text{UT}_n$. Then

- 1) Any left-right orbit of the group $G \times G$ in \mathfrak{ut}_n contains the unique element of the form $X_{D, \phi}$ (denote the orbit by $\mathcal{O}(D, \phi)$).
- 2) Any superclass of the group $G = \text{UT}_n$ contains the unique element of the form $1 + X_{D, \phi}$ (denote the superclass by $K_{D, \phi}$).
- 3) Any left-right orbit of the group $G \times G$ in \mathfrak{ut}_n^* contains the unique element of the form $\lambda_{D, \phi}$ (denote the orbit by $\mathcal{O}^*(D, \phi)$).

The group G decomposes into superclasses $\{K_{D,\phi}\}$, and \mathfrak{ut}_n^* into the left-right orbits $\{\mathcal{O}(D,\phi)\}$. Following the scheme of subsection 3.1, we construct the supercharacter $\chi_{D,\phi}$ by the pair (D,ϕ) . In the papers of C.Andr , these supercharacters are called the *basic characters* and these superclasses are *basic subvarieties* in UT_n . The following theorem is a special case of Theorem 3.1.

Theorem 3.5. The systems of basic characters $\{\chi_{D,\phi}\}$ and basic subvarieties $\{K_{D,\phi}\}$ give rise to a supercharacter theory on the group UT_n .

We correspond the basic subset D to the graph with vertices in $1, 2, \dots, n$; two vertices i and j are connected by a edge if the root (i, j) belongs to D . For example, for $D = \{(1, 3), (3, 6), (2, 4), (4, 5), (5, 7)\}$, the graph has the form:



We say that the roots (i, j) and (k, l) form a *crossing* if $i < k < j < l$. Denote by $c(D)$ the number of crossings in the basic subset D . In the above figure, intersections of edges match crossings of roots; $c(D) = 3$. For any basic subset D , we take

$$d(D) = \sum_{(i,j) \in D} (j - i - 1).$$

The Theorem 3.2 implies the following statement.

Theorem 3.6 [1]. $(\chi_{D,\phi}, \chi_{D,\phi}) = q^{c(D)}$ and $\chi_{D,\phi}(1) = q^{d(D)}$.

If D consists of the only root $\{\alpha\}$ and $c = \phi(\alpha) \in \mathbb{F}_q^*$, then we denote by $\chi_{\alpha,c}$ its supercharacter, and by $\mathcal{O}_{\alpha,c}$ and $\mathcal{O}_{\alpha,c}^*$ the corresponding left-right orbits in \mathfrak{ut}_n and \mathfrak{ut}_n^* . Observe that, in this case, $\chi_{\alpha,c}$ is an irreducible character of the group UT_n (called an *elementary character*), $\mathcal{O}_{\alpha,c}$ is an adjoint orbit, $\mathcal{O}_{\alpha,c}^*$ is a coadjoint orbit (both are called *elementary orbits*).

Theorem 3.7 [3, 4]. Let (D, ϕ) be an admissible pair, $D = \{\alpha_1, \dots, \alpha_k\}$ and $c_i = \phi(\alpha_i)$. Then

- 1) the left-right orbit $\mathcal{O}(D, \phi)$ in \mathfrak{ut}_n is the sum of elementary orbits $\sum_{i=1}^k \mathcal{O}_{\alpha_i, c_i}$;
- 2) the left-right orbit $\mathcal{O}^*(D, \phi)$ in \mathfrak{ut}_n^* is the sum of elementary orbits $\sum_{i=1}^k \mathcal{O}_{\alpha_i, c_i}^*$;
- 3) the supercharacter $\chi_{D,\phi}$ of the group UT_n is the product of elementary characters $\prod_{i=1}^k \chi_{\alpha_i, c_i}$.

The problem of calculation of values of a given supercharacter on superclasses reduces to the case of elementary character. We need new definitions.

Define the operation of partial addition in the set of positive roots by $(i, j) = (i, l) + (l, j)$. Here, we say that the roots (i, l) and (l, j) are *singular* for the root (i, j) . Denote the number of singular roots for α by $\text{Sing}(\alpha)$. For any positive root $\alpha = (i, j)$ and any basic subset D' , denote by $D'(i, j)$ the number of $\beta \in D'$ such that $\text{row}(\beta) > i$ and $\text{col}(\beta) < j$. Take $d'(i, j) = j - i - 1 - |D'(i, j)|$.

Theorem 3.8[4, 1]. Let $\alpha = (i, j)$ be a positive root, $c \in \mathbb{F}_q^*$, and (D', ϕ') is an arbitrary admissible pair. Then the value of the supercharacter $\chi_{\alpha, c}$ on the superclass $K_{D', \phi'}$ can be calculated by the formula

$$\chi_{\alpha, c}(K_{D', \phi'}) = \begin{cases} q^{d'(i, j)} & , \text{ if } D' \cap \text{Sing}(\alpha) = \emptyset, \alpha \notin D'; \\ q^{d'(i, j) \varepsilon^{c\phi'(\alpha)}} & , \text{ if } D' \cap \text{Sing}(\alpha) = \emptyset, \alpha \in D'; \\ 0 & , \text{ if } D' \cap \text{Sing}(\alpha) \neq \emptyset. \end{cases} \quad (13)$$

One can consider the basic subvarieties in \mathfrak{ut}_n^* defined over an arbitrary field. The questions of decomposition of basic subvarieties into the coadjoint orbits and finding generators in the field of invariants are discussed in the papers [6, 31]. Further, we observe the characterization of the basic subvarieties in terms of the tangent cones for the Schubert varieties.

Let K be an arbitrary algebraically closed field. The dual space decomposes into the basic subvarieties $\mathcal{O}_{D, \phi}^*$, where D is a basic subset, and ϕ is a map $D \rightarrow K^*$. A *basic cell* V_D° is a union

$$V_D^\circ = \bigcup_{\phi} \mathcal{O}_{D, \phi}^*;$$

we refer to its closure V_D as a *basic cone*.

Consider the flag variety $\mathcal{F} = G/B$, where $G = \text{GL}(n, K)$ and $B = T(n, K)$. It is well known that \mathcal{F} is decomposed into the Schubert cells $\mathcal{F}_w^\circ = B\dot{w}B \bmod B$, where w is the element of the Weyl group W , and \dot{w} is its representative in the group G . The closure of the Schubert cell \mathcal{F}_w° in the Zariski topology is called a *Schubert variety* \mathcal{F}_w . Any Schubert variety contains the origin point $p = B \bmod B$.

Let N_- be the group of lower triangular matrices of order n with ones on the diagonal. The subset $V = N_-B \bmod B$ is called the *main affine neighborhood* of the point $p \in V$. Applying the presentation $N_- = 1 + \mathfrak{n}_-$, one can identify the main affine neighborhood with \mathfrak{n}_- , and, in its turn, \mathfrak{n}_- , using the Killing form, with \mathfrak{ut}_n^* ; then $p = 0$. Thus, the intersection $\mathcal{F}_w \cap V$ is a Zariski-closed subset in \mathfrak{ut}_n^* . The tangent cone TC_w of the Schubert variety \mathcal{F}_w at the point $p = 0$ is a closed cone in \mathfrak{ut}_n^* with the center at zero.

Definition 3.9. We call the element $w \in W$ *homogeneous* if $\mathcal{F}_w \cap V$ is a cone with the center at the point p (i.e., $\text{TC}_w = \mathcal{F}_w \cap V$).

Theorem 3.10 [31]. The basic cones coincide with the tangent cones of the homogeneous elements of the Weyl group.

3.3 Irreducible constituents of supercharacters

The classification problem of irreducible representations of the unitriangular group reduces to the problem of decomposition of a given supercharacter into a sum of irreducible constituents. The last problem is also extremely difficult in general. However, there are some successes towards a solution.

Observe that the supercharacters $\{\chi_{D,\phi}\}$ with the common D and different $\phi : D \rightarrow \mathbb{F}_q^*$ are conjugate with respect to the adjoint action of the diagonal subgroup $H = \mathbb{F}_q^{*n}$. It is sufficient to obtain a decomposition of only one supercharacter from the series, for example for $\phi \equiv 1$.

First question, when a basic character is irreducible? From Theorem 3.6, the basic character $\chi_{D,\phi}$ is irreducible if and only if $c(D) = 0$, that is, the graph of the basic set has no crossings.

The second question, when a basic character is a multiple irreducible? By a k -crossing, we call the series $i_0 < i_1 < i_2 < \dots < i_k < i_{k+1} < i_{k+2}$, where each pair $(i_s, i_{s+2}) \in D$. The number k stands for the length of the crossing. A maximal crossing is defined in a usual way as a crossing that can't be extended. The following statement can be derived from [7, 4].

Theorem 3.11. The basic character $\chi_{D,\phi}$ is multiple irreducible if and only if each maximal crossing in D has even length. In this case, its unique irreducible component has degree q^e , where $e = d(D) - \frac{1}{2}c(D)$.

In the general case, the supercharacter $\chi_{D,\phi}$ is a linear combination with integer non-negative coefficients of the classfunctions from the set $\text{Kir}(D, \phi)$ that consists of the Kirillov functions associated with the coadjoint orbits from $G\lambda_{D,\phi}G$ [8]. The number of the irreducible characters in $\text{supp}(\chi_{D,\phi})$ coincides with $|\text{Kir}(D, \phi)|$ [27]. One can hypothesize that the sets $\text{supp}(\chi_{D,\phi})$ and $\text{Kir}(D, \phi)$ also coincide, but this is hardly true.

According to Higman's conjecture, the number of irreducible representations of the group $\text{UT}(n, \mathbb{F}_q)$ is a polynomial in q . One can set the similar conjectures for the number $N_D(q)$ of irreducible constituents of the basic character $\chi_{D,\phi}$ and for the number $N_{D,e}(q)$ of its irreducible constituents of degree q^e . This hypotheses are not proved up today. The survey of contemporary results is contained in [26].

In the paper [26], it is verified that the number $N_{D,e}(q)$ can be calculated in terms of irreducible representations of certain algebra $\tilde{\mathfrak{C}}_D(q)$. Denote by $\text{Cr}(D)$ the set of all positive roots (i, j) such that there exist the roots (i, k) , (j, l) from D , where $i < j < k < l$ (i.e., the roots (i, k) and (j, l) produce a

crossing). First, we construct the algebra $\mathfrak{C}_D(q)$ as an algebra generated by the elements e_{ij} , where $(i, j) \in \text{Cr}(D)$, with the relations

$$e_{ij} * e_{kl} = \begin{cases} e_{il}, & \text{if } j = k \text{ and } (i, l) \in \text{Cr}(D), \\ 0, & \text{otherwise} \end{cases}.$$

This multiplication does not coincide with the multiplication of matrix units E_{ij} and E_{kl} . The algebra $\tilde{\mathfrak{C}}_D(q)$ is its central extension $\mathfrak{C}_D(q) \oplus \mathbb{F}_q z_D$ with the multiplication $z_D^2 = 0$, $e_{ij} z_D = z_D e_{ij} = 0$ and

$$e_{ij} * e_{kl} = \begin{cases} e_{il} & , \text{ if } j = k \text{ and } (i, l) \in \text{Cr}(D), \\ z_D & , \text{ if } j = k \text{ and } (i, l) \in D, \\ 0 & , \text{ otherwise} \end{cases}.$$

The algebra $\tilde{\mathfrak{C}}_D(q)$, as $\mathfrak{C}_D(q)$, is associative and nilpotent. Let $J = \mathfrak{ut}_n$. Observe that

$$\mathfrak{C}_D(q) \cong \mathfrak{s}_\lambda / \mathfrak{l}_\lambda \quad \text{and} \quad \tilde{\mathfrak{C}}_D(q) \cong \mathfrak{s}_\lambda / \mathfrak{k}_\lambda,$$

where $\mathfrak{l}_\lambda = J_{\lambda, \text{right}}$, $\mathfrak{s}_\lambda = \{x \in J : \lambda(xy) = 0, \text{ for all } y \in J_{\lambda, \text{right}}\}$, $\mathfrak{k} = J_{\lambda, \text{right}} \cap \text{Ker}(\lambda)$. Let $\text{Irr}(G, k)$ be the set of the irreducible representations of the group G of degree k .

Theorem 3.12 [26, Theorem 3.1]. For any basic subset D and the field \mathbb{F}_q , we obtain the equality

$$N_{D,e}(q) = \frac{\sharp(\text{Irr}(1 + \tilde{\mathfrak{C}}_D(q), q^f) - \sharp(\text{Irr}(1 + \mathfrak{C}_D(q), q^f))}{q - 1}.$$

where $f = c(D) - d(D) + e$. More over, if all irreducible characters of the group $1 + \tilde{\mathfrak{C}}_D(q)$ of degree q^f are the Kirillov functions, then all irreducible constituents in $\chi_{D,\phi}$ of degree q^e are also Kirillov functions.

4 The Hopf algebra of supercharacters for the unitriangular group

4.1 The Hopf algebra $\text{NS}(X)$

Let us recall the definition of the Hopf algebra of symmetric functions in non-commuting variables [1]. Let $X = \{x_1, x_2, \dots\}$ be a set of non-commuting variables (alphabet). The group $S_\infty = \varinjlim S_n$ acts on the set X by finite permutations of the variables. Consider the linear space $\text{NS}(X)$ of all formal power series in variables of X with complex coefficients and bounded degree, and invariant with respect to the group S_∞ . The linear space $\text{NS}(X)$ is a direct sum of the subspaces

$$\text{NS}(X) = \sum_{n=0}^{\infty} \text{NS}_n(X)$$

of all homogeneous elements of the fixed degree. For $n = 0$, the space $\text{NS}_0(X)$ coincides with the field \mathbb{C} . The linear space $\text{NS}(X)$ is a graded algebra with respect to the natural multiplication. The algebra $\text{NS}(X)$ is called the *algebra of symmetric functions in non-commuting variables*.

The algebra $\text{NS}(X)$ is a Hopf algebra defined as follows. Let X' and X'' be two pairwise commuting copies of the alphabet X . Then, for any $f(X) \in \text{NS}(X)$, we have decomposition $f(X' + X'') = \sum f'_k(X') f''_k(X'')$, where $X' + X''$ is a disjoint union of the alphabets X' and X'' . By definition,

$$\Delta(f) = \sum f'_k \otimes f''_k.$$

Let P be a set partition of $[n] = \{1, \dots, n\}$ (denote $P \vdash [n]$). Let m_P be a sum of all monomials $x_{i_1} x_{i_2} \cdots x_{i_n}$, where $x_{i_s} = x_{i_t}$ whenever i_s and i_t belong to a common part of the partition P . For example, of $P = 13|2|4$, then $m_P = x_1 x_2 x_1 x_3 + \dots$

The system of elements $\{m_P : P \vdash [n]\}$ is a basis in $\text{NS}_n(X)$. If $P \vdash [k]$ and $Q \vdash [m]$, then $m_P m_Q \in \text{NS}_n(X)$, $n = k + m$, and

$$m_P m_Q = \sum m_R,$$

where the sum is taken over all $R \vdash [n]$, $R \wedge ([k]||[m]) = (P|Q)$.

By a *subpartition* of the partition $P \vdash [n]$, we call the system of subsets P_1 such that each its subset is one of components of the partition P . By definition, $P = P_1 + P_2$ if the partition P is a disjoint union of two subpartitions P_1 and P_2 .

For any subset $A \subseteq [n]$, denote by st the unique order preserving map $\text{st} : A \rightarrow [|A|]$. Define comultiplication as follows

$$\Delta(m_P) = \sum_{P=P_1+P_2} m_{\text{st}(P_1)} \otimes m_{\text{st}(P_2)}.$$

Example. For the partition $P = 14|2|3|$ of the segment $[4]$, we have $\Delta(m_P) = m_{14|2|3|} \otimes 1 + 2m_{13|2} \otimes m_1 + m_{12} \otimes m_{1|2} + m_{1|2} \otimes m_{12} + 2m_1 \otimes m_{13|2} + 1 \otimes m_{14|2|3|}$.

The linear space $\text{NS}(X)$ is a Hopf algebra with respect to the defined above multiplication, comultiplication, the unit $1 \rightarrow m_\emptyset$ and the counit $f \rightarrow f(0, 0, \dots)$.

4.2 The Hopf algebra SCU

Let $q = 2$ and UT_n be the unitriangular group of the n th order defined over the field of two elements. Consider the linear space of superclass functions SCU_n on the group UT_n and the direct sum

$$\text{SCU} = \sum_{n=1}^{\infty} \text{SCU}_n.$$

Define the multiplication and comultiplication on SCU. The group $UT_k \times UT_m$ is a factor group of UT_n , where $n = k + m$. Let $\phi \in SCU_k$ and $\psi \in SCU_m$, then, by definition,

$$\phi \cdot \psi = \text{Inf}(\phi \times \psi) \in SCU_n,$$

where the inflation is a composition of $\phi \times \psi$ and the natural projection $UT_n \rightarrow UT_k \times UT_m$.

For any partition $T = (A_1|A_2)$ of the set $[n]$, there is the subgroup $UT_{A_1} \times UT_{A_2}$ of the group UT_n ; it consists of all unitriangular matrices such that $a_{ij} \neq 0$ implies that i, j belong to a common component of the partition T . The subgroup $UT_{A_1} \times UT_{A_2}$ is naturally isomorphic to $UT_{|A_1|} \times UT_{|A_2|}$ (denote this isomorphism by π_T). Let $\chi \in SCB_n$. Then, by definition,

$$\Delta(\chi) = \sum_{T=(A_1|A_2) \vdash [n]} {}^T\text{Res}_{UT_{|A_1|} \times UT_{|A_2|}}^{UT_n}(\chi),$$

where ${}^T\text{Res}(\chi)(g) = \text{Res}(\chi)(\pi_T^{-1}(g))$.

We correspond each partition $P \vdash [n]$ to the basic subset D_P as follows. The root (i, j) , where $1 \leq i < j \leq n$, belongs to D_P if i, j are the adjacent elements in a common component of the partition P . For example, if $P = (135|24)$, then $D_P = \{(1, 3), (3, 5), (2, 4)\}$.

Since $q = 2$, we have $\phi \equiv 1$; each partition P corresponds to a unique superclass K_P and supercharacter χ_P . Denote by κ_P the characteristic function on the superclass K_P . The system $\{\kappa_P : P \vdash [n]\}$ is a basis in SCU_n .

Theorem 4.1[1, Theorem 3.2, Corollary 3.3]. The linear space SCU is a Hopf algebra. The map $\kappa_P \rightarrow m_P$, $P \vdash [n]$ can be linearly extended to the isomorphism of the Hopf algebra SCU onto the Hopf algebra $NS(X)$.

Remark 4.2. For the arbitrary q , the Hopf algebra SCU is isomorphic to the "coloured" $NS(X)$ [1, Theorem 3.6].

Remark 4.3. For the arbitrary q , one can consider the supercharacter theory constructed by averaging with respect to the action of group of diagonal matrices Γ (in spirit of example 4 of section 2). Then each partition $P \vdash [n]$ corresponds to the unique supercharacter χ_P^Γ and superclass K_P^Γ . The map that sends the characteristic function κ_P^Γ of the superclass K_P^Γ to the element m_P linearly extends to the isomorphism of the Hopf algebra SCU onto the Hopf algebra $NS(X)$ [9].

4.3 The dual Hopf algebra SCU^*

The linear space $SCU^* = \sum_{n=0}^{\infty} SCU_n^*$ is a Hopf algebra with respect to operations dual to the one in SCU. The scalar product (\cdot, \cdot) on the group

G extends to the scalar product on SCU . Identify SCU^* and SCU as linear spaces; let us calculate the dual operations.

Let $\phi \in \text{SCU}_k$, $\psi \in \text{SCU}_m$ and $n = k + m$. For the partition $T = (A|A^c)$, the operation dual to ${}^T\text{Res}$ is the operation ${}^T\text{SInd} : \text{SCU}_k \otimes \text{SCU}_m \rightarrow \text{SCU}_n$ defined as

$${}^T\text{SInd}(\phi \times \psi) = \text{SInd}(\phi\pi_A \times \psi\pi_A)$$

The multiplication in SCU^* is defined by the formula

$$\phi \cdot \psi = \sum_{T=(A_1|A_2), |A_1|=k, |A_2|=n-k} {}^T\text{SInd}_{\text{UT}_{|A_1|} \times \text{UT}_{|A_2|}}^{\text{UT}_n}(\phi \times \psi). \quad (14)$$

By definition, the comultiplication in SCU^* is the deflation Def that is dual to inflation Inf ; it is defined as follows. Let τ be the natural projection $\text{UT}_n \rightarrow \text{UT}_k \times \text{UT}_m$ and $\chi \in \text{SCU}_n$. Then

$$\Delta(\chi)(u, v) = \text{Def}(\chi)(u, v) = \frac{1}{|\tau^{-1}(1)|} \sum_{x \in \tau^{-1}(u, v)} \chi(x), \quad (15)$$

where $u \in \text{UT}_k$, $v \in \text{UT}_m$.

Theorem 4.4 [1]. The operations of superinduction (14) and deflation (15) give rise to the structure of dual Hopf algebra SCU^* on the linear space SCU .

The system of elements $\{\kappa_P : P \vdash [n]\}$ is a basis in SCU_n . The dual basis $\{\kappa_P^* : P \vdash [n]\}$ consists of the elements $\kappa_P^* = z_P \kappa_P$, where

$$z_P = \frac{|\text{UT}_n|}{|\text{UT}_n X_{D_P} \text{UT}_n|}.$$

Let us calculate the operations of SCU^* on the elements of the basis. For $P \vdash [k]$ and $Q \vdash [m]$, we have

$$\kappa_P^* \cdot \kappa_Q^* = \sum \kappa_{(\text{st}_{A_1}^{-1}(P)|\text{st}_{A_2}^{-1}(Q))}^*,$$

where the sum is taken over all the partitions $(A_1, A_2) \vdash [n]$ such that $|A_1| = k$, $|A_2| = n - k$.

Let $P \vdash [n]$. Then

$$\Delta(\kappa_P^*) = \sum_{k=0}^n \kappa_{P_{[k]}}^* \otimes \kappa_{P_{[k]^c}}^*,$$

where $P_{[k]}$ and $P_{[k]^c}$ are the intersections of the partition P with $[k]$ and its complement $[k]^c$ in $[n]$.

5 Supercharacters of finite solvable groups

5.1 Supercharacter theory for finite groups of triangular type

In this subsection, we construct the supercharacter theory the finite groups of triangular type following the author papers [28, 29, 30].

Let H be a group, and J be an associative algebra defined over a field k . We assume that there are defined the commuting left $h, x \rightarrow hx$ and right $h, x \rightarrow xh$ linear actions of the group H on J . We suppose that, for any $h \in H$ and $x, y \in J$, the following conditions are fulfilled:

1. $h(xy) = (hx)y$ $(xy)h = x(yh)$,
2. $x(hy) = (xh)y$.

The formula

$$g_1 g_2 = (h_1 + x_1)(h_2 + x_2) = h_1 h_2 + h_1 x_2 + x_1 h_2 + x_1 x_2 \quad (16)$$

defines an associative operation on the set

$$G = H + J = \{h + x : h \in H, x \in J\}.$$

If J is a nilpotent algebra over the field k , then G is a group with respect to the operation (16). If the group H is finite, and $k = \mathbb{F}_q$, and J is a finite dimensional nilpotent algebra over k , then the group G is finite.

Definition 5.1. Under the above requirements, we call the group G a *finite group of triangular type* if the group H is abelian and $\text{char } k$ does not divide $|H|$.

Let $G = H + J$ be a finite group of triangular type. The group algebra kH is commutative and semisimple, by Maschke's theorem. Therefore, kH is a sum of fields. There exist the system of primitive idempotents $\{e_1, \dots, e_n\}$ such that

$$kH = k_1 e_1 \oplus \dots \oplus k_n e_n, \quad (17)$$

where k_1, \dots, k_n are extensions of the field k . Any idempotent from kH is a sum of primitive idempotents.

The direct sum $A = kH \oplus J$ has structure of an algebra with respect to the multiplication (16). The group G is a subgroup in the group A^* of invertible elements of the algebra A , see the example 5.4 below. Observe that the group G is decomposed into the product $G = HN$ of its subgroup H and the normal subgroup $N = 1 + J$, which is an algebra group.

Example 5.2. The algebra group $G = 1 + J$.

Example 5.3. $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F}_q, a \neq 0 \right\}$.

Example 5.4. Let A be an associative finite dimensional algebra with unit over the finite field \mathbb{F}_q of q elements [32, §6.6]. By definition, an algebra A is reduced if the its factor algebra over the radical $J = J(A)$ is a direct sum of division algebras. According to Wedderburn's theorem [32, §13.6], any division algebra over a finite field is commutative. Then the algebra A/J is commutative. There exists a semisimple subalgebra S such that $A = S \oplus J$ (see [32, §11.6]). In our case, S is commutative. The group $G = A^*$ on the invertible elements of A is a finite group of triangular type $G = H + J$, where $H = S^*$. If A is the algebra of triangular matrices, then $G = B_n = T(n, \mathbb{F}_q)$ is the triangular group. The supercharacter theory for $G = A^*$ was constructed in [28].

Let $G = H + J$ be a finite group of triangular type. Construct the group \tilde{G} that consists of the triples $\tau = (t, a, b)$, where $t \in H$, $a, b \in N$, with operation

$$(t_1, a_1, b_1) \cdot (t_2, a_2, b_2) = (t_1 t_2, t_2^{-1} a_1 t_2 a_2, t_2^{-1} b_1 t_2 b_2).$$

The group \tilde{G} acts on J by the formula

$$\rho_\tau(x) = t a x b^{-1} t^{-1}. \quad (18)$$

The representation of the group \tilde{G} in the dual space J^* is defined as usual

$$\rho_\tau^* \lambda(x) = \lambda(\rho(\tau^{-1})(x)).$$

In the space J^* , there are defined also the left and right linear actions of the group G by the formulas $b\lambda(x) = \lambda(xb)$ and $\lambda a(x) = \lambda(ax)$. Then $\rho_\tau(\lambda) = t b \lambda a^{-1} t^{-1}$.

For any idempotent $e \in kH$, we denote by A_e the subalgebra eAe . The subalgebra $J_e = eJe \subset J$ is a radical in A_e . Denote $e' = 1 - e$. The decomposition

$$J = eJe \oplus eJe' \oplus e'Je \oplus e'Je'$$

is called the Pierce decomposition. The dual space J_e^* is naturally identified with the subspace in J^* that consists of linear forms equal to zero on all components of Pierce decomposition except for the first component.

Observe that, since the group H is abelian, $he = eh = ehe$ for all $h \in H$. The subset $H_e = eHe$ is a subgroup in the group of all invertible elements of subalgebra A_e . The subgroup $G_e = eGe = H_e + J_e$ is a finite group of triangular type, and this group is associated with the algebra A_e in the same way as G is associated with A . We define the group \tilde{G}_e similar to the group \tilde{G} . The map $h \rightarrow he$ is a homomorphism of the group H onto H_e with the kernel

$$H(e) = \{h \in H : he = e\}. \quad (19)$$

The following definition is equivalent to the corresponding definition from [28, 29], although it differs by a form.

Definition 5.5. An $\rho_{\tilde{G}}$ -orbit \mathcal{O} is *singular* (with respect to H) if $\mathcal{O} \cap J_e \neq \emptyset$ for some idempotent $e \neq 1$ from kH . Otherwise, the orbit \mathcal{O} is *regular* (with respect to H). The elements of the singular (regular) orbits are called singular (regular) elements. Similarly, we define the singular and regular orbits and elements in J^* .

The subgroup $H(e)$ admits the following characterization.

Proposition 5.6 [29, Lemma 2.5]. 1) $H(e) = H_{y,\text{right}} \cap H_{y,\text{left}}$ for any regular (with respect to H_e) element $y \in J_e$.

2) $H(e) = H_{\lambda,\text{right}} \cap H_{\lambda,\text{left}}$ for any regular (with respect to H_e) element $\lambda \in J_e^*$.

In the papers [28, 29], the following statements are proved for any \tilde{G} -orbit \mathcal{O} in J .

Proposition 5.7. 1) For any idempotent $e \in kH$, the intersection $\mathcal{O} \cap J_e$ is a \tilde{G}_e -orbit in J_e .

2) There exists a unique idempotent $e \in kH$ such that $\mathcal{O} \cap J_e$ is a regular \tilde{G}_e -orbit in J_e (with respect to H_e).

Similar statements are true for \tilde{G} -orbits in J^* .

It is known that, for any representation of the finite group in the finite dimensional linear space V defined over a finite field, the number of orbits in V and V^* are equal [16, Lemma 4.1]. This statement and the properties of \tilde{G} -orbits imply that the number of regular (singular) \tilde{G} -orbits in J and J^* are equal [28, Proposition 2.11].

Turn to definition of superclasses in the group G . For any $g \in G$ and $(t, a, n) \in \tilde{G}$ consider the element

$$R_\tau(g) = 1 + ta(g - 1)b^{-1}t^{-1} \quad (20)$$

from the algebra $A = kH + J$. If $g = h + x$, then $R_\tau(g) = h \bmod J$. Therefore, $R_\tau(g) \in G$. The formula (20) defines an action of the group \tilde{G} on G .

Definition 5.8. We refer to the \tilde{G} -orbits in G as *superclasses*.

The group G decomposes into superclasses. Denote by \mathfrak{B} the set of all triples $\beta = (e, h, \omega)$, where e is an idempotent from kH , $h \in H(e)$, and ω is a regular (with respect to H_e) \tilde{G}_e -orbit in J_e . All elements $h + \omega$ are contained in the common superclass [29, corollary 3.2]; we denote it by K_β .

Theorem 5.9 [29, 3.3]. The correspondence $\beta \rightarrow K_\beta$ is a bijection between the set triples \mathfrak{B} and the set of superclasses in G .

Denote by \mathfrak{A} the set of triples $\alpha = (e, \theta, \omega^*)$, where e is an idempotent in kH , θ is a linear character (one dimensional representation) of the group $H(e)$, and ω^* is a regular (with respect to H_e) \tilde{G}_e -orbit in J_e^* . Since the

subgroup $H(e)$ is abelian, then the number of its linear characters equals to the number of elements. The number of regular (with respect to H_e) \tilde{G}_e -orbits in J_e and J_e^* are common. Hence, $|\mathfrak{A}| = |\mathfrak{B}|$.

We turn to construction of supercharacters. Let $\alpha = (e, \theta, \omega^*) \in \mathfrak{A}$, choose $\lambda \in \omega^*$.

Consider the subgroup $G_\alpha = H(e) \cdot N_{\lambda, \text{right}}$, where $N_{\lambda, \text{right}}$ is the stabilizer of λ for the right action of the group $N = 1 + J$ on J^* . The subgroup $N_{\lambda, \text{right}}$ is an algebra subgroup; it is presented in the form $N_{\lambda, \text{right}} = 1 + J_{\lambda, \text{right}}$, where $J_{\lambda, \text{right}}$ is the right stabiliser of λ in J . The group G_α is a sum

$$G_\alpha = H(e) + J_{\lambda, \text{right}}; \quad (21)$$

it is finite group of triangular type.

Fix a nontrivial character $t \rightarrow \varepsilon^t$ of the additive group of the field \mathbb{F}_q with values in the multiplicative group \mathbb{C}^* . By the triple $\alpha = (e, \theta, \omega^*)$ and $\lambda \in \omega^*$, we define the linear character of the group G_α by the formula

$$\xi_{\theta, \lambda}(g) = \theta(h) \varepsilon^{\lambda(x)}, \quad (22)$$

where $g = h + x$, $h \in H(e)$ and $x \in J_{\lambda, \text{right}}$. Let us show that $\xi = \xi_{\theta, \lambda}$ is really a linear character:

$$\begin{aligned} \xi(gg') &= \xi((h + x)(h' + x')) = \xi(hh' + h'x + x'h + xx') = \\ &= \theta(hh') \varepsilon^{\lambda(h'x)} \varepsilon^{\lambda(x'h)} \varepsilon^{\lambda(xx')} = \theta(h) \theta(h') \varepsilon^{\lambda(x)} \varepsilon^{\lambda(x')} = \xi(g) \xi(g'). \end{aligned}$$

The induced character

$$\chi_\alpha = \text{Ind}(\xi_{\theta, \lambda}, G_\alpha, G) \quad (23)$$

is called the *supercharacter*.

Theorem 5.10 [28, 29, Proposition 4.1, 4.4].

- 1) The supercharacters $\{\chi_\alpha : \alpha \in \mathfrak{A}\}$ are pairwise disjoint;
- 2) Each supercharacter χ_α is constant on each superclass K_β ;
- 3) $\{1\}$ is a superclass $K(g)$ for $g = 1$.

Applying the Proposition 2.2, we conclude.

Theorem 5.11 [29, Theorem 4.5]. The systems of supercharacters $\{\chi_\alpha \mid \alpha \in \mathfrak{A}\}$ and superclasses $\{K_\beta \mid \beta \in \mathfrak{B}\}$ give rise to a supercharacter theory on the group G .

Remark. While constructing the supercharacters, it is not easy to find idempotents in the group algebra kH . One can prove [29, section 5] that, in the example 5.4, we may choose idempotents in the subalgebra S (in the case $G = T(n, \mathbb{F}_q)$, the subalgebra S coincides with the subalgebra of diagonal matrices), and, in the example 5.3, we may confine to two idempotents

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the problems of restriction and induction in the constructed supercharacter theory [30]. Let $G = H + J$ be a finite group of triangular type. Let H be a subgroup in H , and J' be a subalgebra in J invariant with respect the left-right action of the group $H' \times H'$ on J . Then $G' = H' + J'$ is a subgroup in G ; we call it the *subgroup of triangular type* in G .

Theorem 5.12 [30]. The restriction of supercharacter of the group G on its subgroup of triangular type is a sum supercharactes of this subgroup with nonnegative integer coefficients.

Let ϕ be a superclass function (i.e., the complex valued function constant on superclasses) on G' . Denote by $\dot{\phi}$ the function on G equal to ϕ on G' and zero outside G' .

Define the superinduction as follows:

$$\text{SInd } \phi(g) = \frac{|H|}{|G| \cdot |G'|} \sum_{\tau \in \tilde{G}} \dot{\phi}(\rho(\tau)(g)) = \frac{|H|}{|G| \cdot |G'|} \sum_{a,b \in N, t \in H} \dot{\phi}(1+ta(g-1)bt^{-1}).$$

Easy to see that $\text{SInd } \phi$ is a superclass function on G . The supercharacter analog of the Frobenius theorem is valid.

Theorem 5.13. Let ψ be a superclass function on G . Then $(\text{SInd } \phi, \psi) = (\phi, \text{Res } \psi)$.

Let $\{\chi_\alpha\}$ be the system of supercharacters for the finite group of triangular type $G = H + J$, and $\{\phi_\eta\}$ be the system of supercharacters of its subgroup of triangular type $G' = H' + J'$. By theorem 5.12,

$$\text{Res } \chi_\alpha = \sum_{\eta} m_{\alpha,\eta} \phi_\eta, \quad m_{\alpha,\eta} \in \mathbb{Z}_+.$$

The system of supercharacters form a basis in the space of superclass functions. Since $\text{SInd } \phi_\eta$ is a superclass function on the group G , we have

$$\text{SInd } \phi_\eta = \sum_{\alpha} a_{\eta,\alpha} \chi_\alpha.$$

The theorem 5.13 implies

$$a_{\eta,\alpha} = \frac{m_{\alpha,\eta}(\phi_\eta, \phi_\eta)}{(\chi_\alpha, \chi_\alpha)}.$$

Corollary 5.14. For any supercharacter ϕ of the subgroup of triangular type G' , the superinduction $\text{SInd } \phi$ is a sum of supercharacters of the group G with nonnegative rational coefficients.

Our next goal is to present the supercharacter version of the A.A.Kirillov formula in the case of the finite groups of triangular type. This formula generates the formulas (10) and (11) for the algebra groups.

Any superclass K finite group of triangular type G has the element of the form $g = h + x$, where $hx = xh = x$. The element $h - 1 \in kH$ is associated with some idempotent $f' \in kH$ (i.e., these elements differ by an invertible multiplier from kH). Let $f = 1 - f'$. The condition $hx = xh = x$ is equivalent to $x \in J_f$. Two elements $g = h + x$ and $g' = h + x'$, where $x, x' \in J_f$, belong to a common superclass whenever x and x' belong to a common \tilde{G}_f -orbit [29, Theorem 3.1].

Since supercharacters are constant on superclasses, it is sufficient calculate the values of supercharacters on the elements of the form $g = h + x$, where $hx = xh = x$. Let $\alpha = (e, \theta, \omega^*)$, and χ_α be a supercharacter. If $h \notin H(e)$, then g does not belong to the Ad_G -orbit of the subgroup G_α . By definition of the induced character, we obtain $\chi_\alpha(h) = 0$.

Consider the case $h \in H(e)$, i.e., $he = e$; then $f'e = 0$ and $e < f$. We see $J_e \subset J_f$, and J_e^* is a subspace in J_f^* . In particular, $\omega^* \in J_f^*$. There exists a unique $\rho^*(\tilde{G}_f)$ -orbit Ω^* in J_f^* such that its intersection with J_e^* coincides with ω^* (see Proposition 5.7). Denote by $\dot{\theta}(h)$ the function $H \rightarrow \mathbb{C}$ equal to $\theta(h)$ on $H(e)$ and zero outside $H(e)$.

Theorem 5.15. The value of the supercharacter χ_α on the element $g = h + x$, where $hx = xh = x$, can be calculated by the formula

$$\chi_\alpha(g) = \frac{|H_e| \cdot \dot{\theta}(h)}{n(\Omega^*)} \sum_{\mu \in \Omega^*} \varepsilon^{\mu(x)}, \quad (24)$$

where $n(\Omega^*)$ is the number of right N_f -orbits in Ω^* .

The other form of A.A.Kirillov formula has the form.

Theorem 5.16. The value of the supercharacter χ_α on the element $g = h + x$, where $hx = xh = x$, can be calculated by the formula

$$\chi_\alpha(g) = \frac{|H_e| \cdot |\lambda N_f| \cdot \dot{\theta}(h)}{|\rho(\tilde{G}_f)(x)|} \sum_{y \in \rho(\tilde{G}_f)(x)} \varepsilon^{\lambda(y)}. \quad (25)$$

The next goal is to obtain the group of triangular type version of the theorem 3.2. The scalar product on the group G is defined as in (12).

Lemma 5.17. Let $\lambda \in J_f^*$. Then $J_f \lambda \cap \lambda J_f = J \lambda \cap \lambda J$.

Proof. Since $\lambda \in J_f^*$, we have $\lambda f = f \lambda = \lambda$. The inclusion $J_f \lambda \cap \lambda J_f \subseteq J \lambda \cap \lambda J$ is obvious. Let $\mu \in J \lambda \cap \lambda J$. Then $\mu = j_1 \lambda$ and $\mu = \lambda j_2$ for some $j_1, j_2 \in J$. Hence, $\mu f = j_1 \lambda f = j_1 \lambda = \mu$ and $f \mu = f \lambda j_2 = \lambda j_2 = \mu$. Therefore,

$$\mu = f \mu f \in f(J \lambda \cap \lambda J) f = (f J f)(f \lambda f) \cap (f \lambda f)(f J f) = J_f \lambda \cap \lambda J_f.$$

□

Theorem 5.18. Let $\alpha = (e, \theta, \omega^*) \in \mathfrak{A}$ and $\lambda \in \omega^*$. Then

$$(\chi_\alpha, \chi_\alpha) = \frac{|H_{N\lambda N}|}{|H(e)|} \cdot |J\lambda \cap \lambda J|, \quad (26)$$

where $H_{N\lambda N}$ is the stabilizer of $N\lambda N$ with respect to Ad_H^* action the subgroup H .

Proof. The subgroup $G' = H(e)N$ contains G_α . Consider the character $\chi'_{\theta, \lambda} = \text{Ind}(\xi_{\theta, \lambda}, G_\alpha, G')$ of the subgroup G' . Applying the Intertwining Number Theorem [24, 44.5], we obtain

$$(\chi_\alpha, \chi_\alpha) = (\text{Ind}(\chi'_{\theta, \lambda}, G', G), \text{Ind}(\chi'_{\theta, \lambda}, G', G)) = \sum_{h \in H/H(e)} (\chi'_{\theta, \text{Ad}_h^* \lambda}, \chi'_{\theta, \lambda}),$$

The subgroup G' is also a finite subgroup of triangular type; its diagonal part $H(e)$ identically acts on λ . Two characters $\chi'_{\theta, \text{Ad}_h^* \lambda}$ $\chi'_{\theta, \lambda}$ are disjoint if $\text{Ad}_h^* \lambda \notin N\lambda N$. Therefore,

$$(\chi_\alpha, \chi_\alpha) = \frac{|H_{N\lambda N}|}{|H(e)|} (\chi'_{\theta, \lambda}, \chi'_{\theta, \lambda}). \quad (27)$$

We apply the formula (24) to the supercharacter $\chi'_{\theta, \lambda}$ of the group G' . For $g = h + y \in G'$, $hy = yh = y$, we have

$$\chi'_{\theta, \lambda}(g) = \theta(h) \chi_{\theta, \lambda}^f(1 + y),$$

where $\chi_{\theta, \lambda}^f$ is a supercharacter of the group N_f constructed by $\lambda \in J_e^* \subseteq J_f^*$.

Fix $h \in H(e)$ and $x \in J$ such that $hx = xh = x$. As above, the condition on x is equivalent to $x \in J_f$, where $f = 1 - f'$ and the idempotent f' is associated to $h - 1$. Each element of the superclass $K(h + x)$ in G' can be uniquely presented in the form $(1 + v)(h + y)(1 + u)$, where $v \in fJf'$, $u \in f'J$, and $y \in N_fxN_f$. The supercharacter is constant on superclasses; then

$$\sum_{g \in K(h+x)} \chi'_{\theta, \lambda}(g) \overline{\chi'_{\theta, \lambda}(g)} = \frac{|J|}{|J_f|} \sum_{y \in N_fxN_f} \chi_{\theta, \lambda}^f(1 + y) \overline{\chi_{\theta, \lambda}^f(1 + y)}.$$

Hence

$$\begin{aligned} \sum_{g \bmod J=h} \chi'_{\theta, \lambda}(g) \overline{\chi'_{\theta, \lambda}(g)} &= \frac{|J|}{|J_f|} \sum_{y \in J_f} \chi_{\theta, \lambda}^f(1 + y) \overline{\chi_{\theta, \lambda}^f(1 + y)} = \\ &|J| \cdot (\chi_{\theta, \lambda}^f, \chi_{\theta, \lambda}^f) = |J| \cdot |J_f \lambda \cap \lambda J_f| = |J| \cdot |J\lambda \cap \lambda J|. \end{aligned} \quad (28)$$

Observe that the sum (28) does not depend on $h \in H(e)$. Then

$$\begin{aligned} (\chi_\alpha, \chi_\alpha) &= \frac{|H_{N\lambda N}|}{|H(e)|} \cdot \frac{1}{|G'|} \sum_{g \in G'} \chi'_\alpha(g) \overline{\chi'_\alpha(g)} = \\ &= \frac{|H_{N\lambda N}|}{|H(e)| \cdot |G'|} \cdot |H(e)| \cdot |J| \cdot |J\lambda \cap \lambda J| = \frac{|H_{N\lambda N}|}{|H(e)|} \cdot |J\lambda \cap \lambda J|. \quad \square \quad (29) \end{aligned}$$

5.2 The supercharacter theory for the triangular group

Consider the algebra $A = \mathfrak{t}(n, \mathbb{F}_q)$ that consists of all $n \times n$ -matrices with entries from the field \mathbb{F}_q and zeros below the diagonal. The *triangular group* $G = B_n = T(n, \mathbb{F}_q)$ is a group of all invertible elements in A ; it is a finite group of triangular type $G = H + J$, where J is a subalgebra $\mathfrak{ut}_n = \mathfrak{ut}(n, \mathbb{F}_q)$ of all triangular matrices with zeros on the diagonal, and $H = \{(a_1, \dots, a_n) : a_i \in \mathbb{F}_q^*\}$ is the subgroup of diagonal matrices. The subgroup G is a semidirect product $G = HN$, where $N = UT_n$. In this subsection, we concretize the constructed supercharacter theory to the triangular group.

As above, a positive root is a pair (i, j) , where $1 \leq i < j \leq n$. Let D be a basic subset (see subsection 3.2). By a *support* of D , we call the subset $\text{supp}(D) = \text{row}(D) \cup \text{col}(D)$ in $[n]$. For each basic subset D , we construct the element

$$x_D = \sum_{(i,j) \in D} E_{ij}$$

in J . Denote by \mathcal{O}_D the orbit of the element x_D with respect to the action ρ of the group \tilde{G} on J (see (18)). Each \tilde{G} -orbit in J has the form \mathcal{O}_D for some basic subset D . Similarly for J^* ; each \tilde{G} -orbit in J^* is the orbit \mathcal{O}_D^* of some

$$\lambda_D = \sum_{(i,j) \in D} E_{ij}^*.$$

Lemma 5.19 [28]. The orbit \mathcal{O}_D (respectively, \mathcal{O}_D^*) is regular if and only if $\text{supp}(D) = [1, n]$.

By basic subset D , we construct idempotent

$$e = e_D = \sum_{i \in \text{supp}(D)} E_{ii}.$$

The subgroup H decomposes into the product of subgroups $H_i = \{(1, \dots, a_i, \dots, 1)\}$ isomorphic to \mathbb{F}_q^* . The subgroups $H(e)$ and H_e , defined in (19), have the form

$$H_e = \prod_{i \in \text{supp}(D)} H_i, \quad H(e) = \prod_{i \in [n] \setminus \text{supp}(D)} H_i.$$

The subgroup H is the product $H = H_e \cdot H(e)$.

For the idempotent $e = e_D$, the subgroup G_e is naturally isomorphic to triangular subgroup B_m , where m is the number of elements in $\text{supp}(D)$. The element x_D belongs to $J_e = eJ_e$ (respectively, λ_D belongs to J_e^*). Denote by ω_D the \tilde{G}_e -orbit of x_D in J_e . Respectively, ω_D^* is the \tilde{G}_e -orbit of λ_D in J_e^* . By Lemma 5.19, we obtain that ω_D and ω_D^* are regular orbits in J_e and J_e^* (with respect to H_e).

Example. $G = T(3, \mathbb{F}_q) = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$, and $D = \{(1, 3)\}$. Then

$$e = e_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H_e = \begin{pmatrix} * & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & * \end{pmatrix}, \quad H(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$J_e = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_e = \begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & * \end{pmatrix},$$

$$\omega_D = \begin{pmatrix} 0 & 0 & \neq 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is a regular } \tilde{G}_e\text{-orbit in } J_e.$$

Given the basic subset D and the element $h \in H(e)$, we construct the superclass $K_{h,D'}$ as the \tilde{G} -orbit of the element $h + x_D$ in the group G . For D and the character θ of the group $H(e)$, we define the supercharacter $\{\chi_{\theta,D}\}$ of the group G as in the subsection 5.1 for the triple (e, θ, ω) .

Theorem 5.20 [28]. The systems of supercharacters $\{\chi_{\theta,D}\}$ and superclasses $\{K_{h,D}\}$ give rise to a supercharacter theory of the group $G = B_n$.

Let us calculate the values of the supercharacter $\chi_{\theta,D}$ on the superclass $K_{h,D'}$ of the element $h + x_{D'}$. We need some new notations.

For each positive root $\gamma = (i, j)$, $1 \leq i < j \leq n$, we denote

$$\Delta'(\gamma) = \{(i, k) \mid i < k < j\}, \quad \Delta''(\gamma) = \{(k, j) \mid i < k < j\}.$$

The numbers of elements of both subsets are equal and coincide with $j - i - 1$. We take $\delta'(D, D') = 0$ if there exists $\gamma \in D$ and $\gamma \in D'$ such that $\gamma' \in \Delta'(\gamma)$. Otherwise, we take $\delta'(D, D') = 1$. Similarly, we define $\delta''(D, D')$.

Take $\delta_0(D, h) = 1$ if $h \in H(e_D)$. Otherwise, $\delta_0(D, h) = 0$. Denote

$$\delta(D, h, D') = \delta'(D, D')\delta''(D, D')\delta_0(D, h). \quad (30)$$

For any positive root $\gamma = (i, j)$, $1 \leq i < j \leq n$, we denote by $P(\gamma)$ the submatrix of the matrix $h - 1 + x_{D'}$ with systems of rows and columns

$[i + 1, j - 1]$. Let $m(\gamma, h, D')$ denote corank of $P(\gamma)$. Since there is at most one element in each row and column of P_γ , $m(\gamma, h, D')$ is the number of zero rows (columns) in P_γ . Introduce the notations

$$m(D, h, D') = \sum_{\gamma \in D} m(\gamma, h, D'), \quad s(D, D') = |D| + |D \setminus D'|. \quad (31)$$

Theorem 5.21. The value of the supercharacter $\chi_{\theta, D}$ on the superclass $K_{h, D'}$ equals to

$$\chi_{\theta, D}(K_{h, D'}) = \delta(D, h, D')(-1)^{|D \cap D'|} q^{m(D, h, D')} (q - 1)^{s(D, D')} \theta(h). \quad (32)$$

6 The Hopf algebra of supercharacters of the triangular group

6.1 The Hopf algebra $\text{NPS}(X, Y)$

Let we have the system of non-commuting variables $X \cup Y$ that consists of two parts $X = \{x_1, x_2, \dots\}$ and $Y = \{y_1, \dots, y_{|Y|}\}$. As in subsection 4.1, the group $S_\infty = \varinjlim S_n$ acts on the set X by finite permutations. Extend this action to the set $X \cup Y$ setting it identical on Y . Consider the linear space $\text{NPS}(X, Y)$ of all formal complex power series in variables of $X \cup Y$ of bounded degree in X and finite degree in Y , and invariant with respect to the group S_∞ . The linear space $\text{NPS}(X, Y)$ is a direct sum of the subspaces

$$\text{NPS}(X, Y) = \sum_{n=0}^{\infty} \text{NPS}_n(X, Y)$$

of homogeneous elements of fixed degree. For $n = 0$, the subspace $\text{NPS}_0(X, Y)$ coincides with the field \mathbb{C} .

The linear space $\text{NPS}(X, Y)$ is a graded algebra under the natural multiplication. We call the algebra $\text{NPS}(X, Y)$ the *algebra of partially symmetric functions in non-commuting variables*.

Let A be a finite subset of positive integers.

Definition 6.1. By a *rigged partition* of A we call a pair $\mathcal{P} = (P, \Phi_{\mathcal{P}})$, where P is a partition of some subset $\text{supp}(P) \subseteq A$ (the partial partition of the set A), and $\Phi_{\mathcal{P}}$ is a map $A \setminus \text{supp}(P) \rightarrow [|Y|]$. Denote $\mathcal{P} \models A$.

Definition 6.2. Let $\mathcal{P} \models A$ and $\mathcal{P} = (P, \Phi_{\mathcal{P}})$. Then $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, where $\mathcal{P}_i \models A_i$, $\mathcal{P}_i = (P_i, \Phi_{\mathcal{P}_i})$, if $A = A_1 \sqcup A_2$, $P = P_1 + P_2$ (see subsection 4.1), the map $\Phi_{\mathcal{P}}$ coincides with $\Phi_{\mathcal{P}_i}$ being reduced on $A_i \setminus \text{supp}(\mathcal{P}_i)$.

Let $\mathcal{P} = (P, \Phi_{\mathcal{P}})$ be a ridded partition of the set $[n]$. Enumerate the components of the partial partition $P = \{\Pi_1, \dots, \Pi_s\}$. Correspond \mathcal{P} to the element $m_{\mathcal{P}} \in \text{NPS}_n(X, Y)$ that is a sum of all monomials of the form $\sigma(z_1 \cdots z_n)$, where $\sigma \in S_\infty$, $z_i = y_{\Phi(i)}$ for all $i \in [n] \setminus \text{supp}(P)$, and $z_i = x_j$

for $i \in \Pi_j$. The element $m_{\mathcal{P}}$ does not depend on the enumeration of partial partition P .

Example. $n = 4$, $P = 13|4$ $\Phi(2) = 1$. Then $m_{\mathcal{P}}$ is the sum $x_1y_1x_1x_2 + x_2y_1x_2x_1 + x_1y_1x_1x_3 + \dots$

The system of elements $\{m_{\mathcal{P}} : \mathcal{P} \models [n]\}$ is a basis in $\text{NPS}_n(X, Y)$. For $\mathcal{P} \models [k]$ and $\mathcal{Q} \models [n - k]$, we consider the rigged partition $\mathcal{S} = (S, \Phi_{\mathcal{S}})$, where $S = P \sqcup \{k + Q\}$, and $\Phi_{\mathcal{S}}(i) = \Phi_{\mathcal{P}}(i)$ if $1 \leq i \leq k$, and $\Phi_{\mathcal{S}}(k + j) = \Phi_{\mathcal{Q}}(j)$ if $1 \leq j \leq n - k$. Denote $\mathcal{S} = (\mathcal{P}|\mathcal{Q})$.

We say that the rigged partition $\mathcal{R} = (R, \Phi_{\mathcal{R}})$ is a *direct consequence* of the partition $(\mathcal{P}|\mathcal{Q})$ if $\text{supp}(R) = \text{supp}(P) \sqcup \text{supp}(Q)$, $R \wedge ([k][n - k]) = (P|Q)$, and $\Phi_{\mathcal{R}}$ coincides with $\Phi_{(\mathcal{P}|\mathcal{Q})}$. Denote $(\mathcal{P}|\mathcal{Q}) \rightarrow \mathcal{R}$. Then

$$m_{\mathcal{P}}m_{\mathcal{Q}} = \sum_{(\mathcal{P}|\mathcal{Q}) \rightarrow \mathcal{R}} m_{\mathcal{R}}. \quad (33)$$

We define a comultiplication by the formula

$$\Delta(m_{\mathcal{P}}) = \sum_{\mathcal{P}=\mathcal{P}_1+\mathcal{P}_2} m_{\text{st}(\mathcal{P}_1)} \otimes m_{\text{st}(\mathcal{P}_2)}. \quad (34)$$

The counit is defined as the map that corresponds the series to its constant term.

Theorem 6.3. The algebra $\text{NPS}(X, Y)$ is a graded Hopf algebra under the defined comultiplication and counit.

Proof. Since the space $\text{NPS}(X, Y)$ is a connected graded algebra (i.e., $\text{NPS}_0(X, Y) = \mathbb{C}$), it is sufficient to prove that $\text{NPS}(X, Y)$ is a bialgebra ([36, 18]), that is, the comultiplication is an algebra homomorphism. For $\mathcal{P} \models [k]$ and $\mathcal{Q} \models [n - k]$, applying (33) and (34), we obtain

$$\begin{aligned} \Delta(m_{\mathcal{P}})\Delta(m_{\mathcal{Q}}) &= \left(\sum_{\mathcal{P}=\mathcal{P}_1+\mathcal{P}_2} m_{\text{st}(\mathcal{P}_1)} \otimes m_{\text{st}(\mathcal{P}_2)} \right) \left(\sum_{\mathcal{Q}=\mathcal{Q}_1+\mathcal{Q}_2} m_{\text{st}(\mathcal{Q}_1)} \otimes m_{\text{st}(\mathcal{Q}_2)} \right) = \\ &= \sum_{\mathcal{P}=\mathcal{P}_1+\mathcal{P}_2, \mathcal{Q}=\mathcal{Q}_1+\mathcal{Q}_2} m_{\text{st}(\mathcal{P}_1)}m_{\text{st}(\mathcal{Q}_1)} \otimes m_{\text{st}(\mathcal{P}_2)}m_{\text{st}(\mathcal{Q}_2)} = \sum_{\mathcal{R}=\mathcal{R}_1+\mathcal{R}_2, (\mathcal{P}|\mathcal{Q}) \rightarrow \mathcal{R}} m_{\text{st}(\mathcal{R}_1)} \otimes m_{\text{st}(\mathcal{R}_2)}. \end{aligned}$$

On the other hand,

$$\Delta(m_{\mathcal{P}}m_{\mathcal{Q}}) = \Delta \left(\sum_{(\mathcal{P}|\mathcal{Q}) \rightarrow \mathcal{R}} m_{\mathcal{R}} \right) = \sum_{\mathcal{R}=\mathcal{R}_1+\mathcal{R}_2, (\mathcal{P}|\mathcal{Q}) \rightarrow \mathcal{R}} m_{\text{st}(\mathcal{R}_1)} \otimes m_{\text{st}(\mathcal{R}_2)}.$$

Hence $\Delta(m_{\mathcal{P}}m_{\mathcal{Q}}) = \Delta(m_{\mathcal{P}})\Delta(m_{\mathcal{Q}})$. \square

6.2 The Hopf algebra SCB

Let $B_n = \text{UT}(n, \mathbb{F}_q)$. Consider the space of superclass functions SCB_n on the group B_n and the direct sum

$$\text{SCB} = \sum_{n=0}^{\infty} \text{SCB}_n.$$

The multiplication and comultiplication in SCB are defined below similarly to the algebra SCU.

The group $B_k \times B_m$ is a factor group of B_n , where $n = k + m$. Let $\phi \in \text{SCB}_k$ and $\psi \in \text{SCB}_m$, then, by definition,

$$\phi \cdot \psi = \text{Inf}(\phi \times \psi) \in \text{SCB}_n, \quad (35)$$

where the inflation is a composition of $\phi \times \psi$ and the natural projection $B_n \rightarrow B_k \times B_m$.

For any partition $T = (A_1|A_2)$ of the set $[n]$, there exists the subgroup $B_{A_1} \times B_{A_2}$ of the group B_n ; the subgroup consists of all unitriangular matrices such that $a_{ij} \neq 0$ implies that i, j lie in a common component of the partition T . The subgroup $B_{A_1} \times B_{A_2}$ is naturally isomorphic to $B_{|A_1|} \times B_{|A_2|}$ (denote the isomorphism by π_T). Let $\chi \in \text{SCB}_n$. Then, by definition,

$$\Delta(\chi) = \sum_{T=(A_1|A_2) \vdash [n]} {}^T \text{Res}_{B_{|A_1|} \times B_{|A_2|}}^{B_n}(\chi), \quad (36)$$

where

$${}^T \text{Res}_{B_{|A_1|} \times B_{|A_2|}}^{B_n}(\chi)(g) = \text{Res}_{B_{|A_1|} \times B_{|A_2|}}^{B_n}(\chi)(\pi_T^{-1}(g)).$$

Suppose that $|Y| = q - 2$. Fix enumerations in the set $\mathbb{F}_q^* \setminus \{1\} = \{u_1, \dots, u_{q-2}\}$ and in set $\text{Irr}(\mathbb{F}_q^*) \setminus \{1\} = \{\eta_1, \dots, \eta_{q-2}\}$. We correspond the rigged partition $\mathcal{P} = (P, \Phi_{\mathcal{P}}) \vdash [n]$ to the supercharacter as follows. By the partial partition P , we construct the basic subset D_P in the set of positive roots, as in subsection 4.2. Observe that $\text{supp}(P) \subseteq \text{supp}(D)$ and the equality holds if the partition P has no singleton components.

Construct the element

$$h_{\mathcal{P}} = \prod_{i \in [n] \setminus \text{supp}(P)} u_{\Phi_{\mathcal{P}}(i)} \in H(e_{D_P})$$

and the superclass $K_{\mathcal{P}}$ of the element $h_{\mathcal{P}} + x_{D_P}$. We construct also the supercharacter $\chi_{\mathcal{P}}$ by the subset D_P and by the character

$$\theta_{\mathcal{P}} = \prod_{i \in [n] \setminus \text{supp}(P)} \eta_{\Phi_{\mathcal{P}}(i)}$$

of the subgroup $H(e_{D_P})$. In terms of rigged partitions, Theorem 5.20 can be presented in the following way.

Proposition 6.4. The systems of supercharacters $\{\chi_{\mathcal{P}}\}$ and superclasses $K_{\mathcal{P}}$, where \mathcal{P} runs through the set of all rigged partitions of $[n]$, give rise to a supercharacter theory of the triangular group B_n .

We denote by $\kappa_{\mathcal{P}}$ the characteristic function of the superclass $K_{\mathcal{P}}$. Let as above $|Y| = q - 2$.

Theorem 6.5. The linear space SCB is a Hopf algebra with respect to the defined multiplication and comultiplication. The map $\kappa_{\mathcal{P}} \rightarrow m_{\mathcal{P}}$, where \mathcal{P} runs through the set of all rigged partitions of $[n]$, can be linearly extended to the isomorphism of the Hopf algebra SCB onto the Hopf algebra $NPS(X, Y)$.

Proof. It is sufficient to prove the multiplication and comultiplication of the elements $\kappa_{\mathcal{P}}$ satisfies the equalities (33) and (34).

The multiplication is defined by the formula (35). Let $e_{[k]}$ be the idempotent equal to the sum of first k diagonal matrix units, and $e_{[n-k]}$ be the sum of the last $n - k$ ones. Then $\kappa_{\mathcal{P}}\kappa_{\mathcal{Q}}$ is the characteristic function of the set

$$\Lambda = \text{diag}(K_{\mathcal{P}}|K_{\mathcal{Q}}) + e_{[k]}Je_{[n-k]}.$$

Denote $G = B_n$. The set Λ is invariant with respect to the action of the group \tilde{G} and, therefore, it decomposes into superclasses (see subsection 5.1). All elements from Λ has common diagonal part $h = \text{diag}(h_{\mathcal{P}}, h_{\mathcal{Q}})$. Let f' be the idempotent associated with the diagonal matrix $h - 1$. For the idempotent $f = 1 - f'$, there defined the subgroup $G_f = H_f + J_f$, where $H_f = fHf$ and $J_f = fJf$. The subgroup G_f is isomorphic to B_s for some $s \leq n$; it is a semidirect product $G_f = H_fN_f$, where $N_f = 1 + J_f$ is isomorphic to UT_s . Similarly for $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$ one can construct the idempotents $f_{\mathcal{P}} \leq e_{[k]}$ and $f_{\mathcal{Q}} \leq e_{[n-k]}$. Then $f = f_{\mathcal{P}} + f_{\mathcal{Q}}$; N_f contains the subgroup $N_{f_{\mathcal{P}}} \times N_{f_{\mathcal{Q}}}$ isomorphic to $UT_{s_1} \times UT_{s_2}$, where $s = s_1 + s_2$.

Each superclass of element with the diagonal part h contains the subset $h + \omega$, where ω is some orbit in J_f with respect to the group \tilde{G}_f (see subsection 5.1). So, the set $1 + \omega$ is the H_f -orbit in the set of superclasses of the group $N_f = UT_s$; it is uniquely determined by the given superclass with diagonal part h . If, in addition, this superclass belongs to Λ , then the projection of $1 + \omega$ on $UT_{s_1} \times UT_{s_2}$ coincides with the H_f -orbit of the superclass of the element $\text{diag}(1 + x_{D_P}, 1 + x_{D_Q})$. Applying statements of subsection 4, $1 + \omega$ corresponds to the partition R of the set $[s]$ and $R \wedge ([s_1]||[s_2]) = (P|Q)$ (see Remark 4.3). The superclasses that belong to Λ are the superclasses of the group B_n of the form $K_{\mathcal{R}}$, where $(\mathcal{P}|\mathcal{Q}) \rightarrow \mathcal{R}$. This proves

$$\kappa_{\mathcal{P}}\kappa_{\mathcal{Q}} = \sum_{(\mathcal{P}|\mathcal{Q}) \rightarrow \mathcal{R}} \kappa_{\mathcal{R}}. \quad (37)$$

According to the formula (34), $\Delta(\kappa_{\mathcal{P}})$ is the sum of the restriction of $\kappa_{\mathcal{P}}$ on the subgroups $B_{A_1} \otimes B_{A_2}$. Each restriction is the characteristic function of the intersection of the superclass $K_{\mathcal{P}}$ with the subgroup $B_{A_1} \otimes B_{A_2}$ isomorphic to $B_{|A_1|} \times B_{|A_2|}$. This intersection decomposes into superclasses $\{K_{\mathcal{P}_1} \times K_{\mathcal{P}_2}\}$ of the group $B_{A_1} \otimes B_{A_2}$. The \tilde{G} -orbit of each superclass $K_{\mathcal{P}_1} \times K_{\mathcal{P}_2}$ coincides with $K_{\mathcal{P}}$. This proves that the intersection consists of a single superclass and $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$. Hence

$$\Delta(\kappa_{\mathcal{P}}) = \sum_{\mathcal{P}=\mathcal{P}_1+\mathcal{P}_2} \kappa_{\text{st}(\mathcal{P}_1)} \otimes \kappa_{\text{st}(\mathcal{P}_2)}. \quad \square$$

6.3 The dual Hopf algebra SCB^*

The linear space $\text{SCB}^* = \sum_{n=0}^{\infty} \text{SCB}_n^*$ is a Hopf algebra with respect to operations dual to operations in SCB . By the scalar product, we identify SCB^* and SCB as a linear spaces. The definition of superinduction for the groups of triangular type enables to calculate the dual operations in term of the space SCB (see Theorem 5.13).

Let $\phi \in \text{SCB}_k$, $\psi \in \text{SCB}_m$ and $n = k + m$. For the partition $T = (A|A^c)$ the dual operation to ${}^T\text{Res}$ is the operation ${}^T\text{SInd} : \text{SCB}_k \otimes \text{SCB}_m \rightarrow \text{SCB}_n$ defined as

$${}^T\text{SInd}(\phi \times \psi) = \text{SInd}(\phi \pi_A \times \psi \pi_A).$$

The multiplication in SCB^* is defined by the formula

$$\phi \cdot \psi = \sum_{T=(A_1|A_2), |A_1|=k, |A_2|=n-k} {}^T\text{SInd}_{B_{|A_1|} \times B_{|A_2|}}^{B_n}(\phi \times \psi). \quad (38)$$

By definition, comultiplication in SCB^* is the deflation Def that is a dual operation to the inflation Inf . Let τ be the natural projection $B_n \rightarrow B_k \times B_m$ and $\chi \in \text{SCB}_n$. Then

$$\text{Def}(\chi)(u, v) = \frac{1}{|\tau^{-1}(1)|} \sum_{x \in \tau^{-1}(u, v)} \chi(x), \quad (39)$$

where $u \in B_k$, $v \in B_m$.

Theorem 6.6. The operations (38) and (39) define the structure of dual Hopf algebra SCB^* on the linear space SCB .

The system $\{\kappa_{\mathcal{P}} : \mathcal{P} \models [n]\}$ is a basis in SCB_n . The dual basis $\{\kappa_{\mathcal{P}}^* : \mathcal{P} \models [n]\}$ consists of the elements $\kappa_{\mathcal{P}}^* = z_{\mathcal{P}} \kappa_{\mathcal{P}}$, where $z_{\mathcal{P}} = |B_n|/|K_{\mathcal{P}}|$. Calculate the operations of SCB^* on the basic elements. For $\mathcal{P} \models [k]$ and $\mathcal{Q} \models [m]$ we have

$$\kappa_{\mathcal{P}}^* \cdot \kappa_{\mathcal{Q}}^* = \sum \kappa_{(\text{st}_{A_1}^{-1}(\mathcal{P})|\text{st}_{A_2}^{-1}(\mathcal{Q}))}^*,$$

where the sum is taken over all partitions $(A_1, A_2) \vdash [n]$ such that $|A_1| = k$, $|A_2| = n - k$.

Let $\mathcal{P} \models [n]$. Then

$$\Delta(\kappa_{\mathcal{P}}^*) = \sum_{k=0}^n \kappa_{\mathcal{P}_{[k]}}^* \otimes \kappa_{\mathcal{P}_{[k]^c}}^*,$$

where $\mathcal{P}_{[k]}$ and $\mathcal{P}_{[k]^c}$ are the intersections of the rigged partition \mathcal{P} with $[k]$ and its complement $[k]^c$ in $[n]$.

References

- [1] Aguiar M., Andr  C., Benedetti C., Bergeron N., Zhi Chen, Diaconis P., Hendrickson A., Hsiao S., Isaacs I.M., Jedwab A., Johnson K., Karaali G., Lauve A., Tung Le, Lewis S., Huilan Li, Magaarg K., Marberg E., Novelli J-Ch., Amy Pang, Saliola F., Tevlin L., Thibon J-Y., Thiem N., Venkateswaran V., Vinroot C.R., Ning Yan, Zabricki M., *Supercharacters, symmetric functions in noncommuting variables, and related Hopf algebras*, Advances in Mathematics, 2012, vol. 229, no.4, 2310-2337.
- [2] Andr  C.A.M., *Basic characters of the unitriangular group*, J. Algebra, 1995, vol. 175, 287-319.
- [3] Andr  C.A.M., *Basic sums of coadjoint orbits of the unitriangular group*, 1995, J. Algebra, vol. 176, 959-1000.
- [4] Andr  C.A.M., *The basic character table of the unitriangular group*, J. Algebra, 2001, vol. 241, 437-471.
- [5] Andr  C.A.M., *The basic characters of the unitriangular group (for arbitrary primes)*, Proc. Am. Math. Soc., 2002, vol. 130, 1943-1954.
- [6] Andr  C.A.M., *Basic sums of Coadjoint Orbits of the Unitriangular group*, J. Algebra, 1995, vol. 176, 959-1000.
- [7] Andr  C.A.M., *Hecke algebras for the basic characters of the unitriangular group*, Proc.Amer.Math.Soc., 2003, vol.132, 987-996.
- [8] Andr  C.A.M., Nicol s A., *Supercharacters of the adjoint group of a finite radical ring*, J.Group Theory, 2008, vol. 11, 709-746.
- [9] Andr  C.A.M., *Supercharacters of unitriangular groups and set-partition combinatorics*, CIMPA school: Modern Methods in Combinatorics ECOS 2013, San Luis, Argentina.

- [10] André C.A.M., Neto A.M., *A supercharacter theory for the Sylow p -subgroups of the finite symplectic and orthogonal groups*, J. Algebra, vol. 322, no. 4, 2009, 1273-1294.
- [11] Arias-Castro E., Diaconis P., Stanley R., *A super-class walk on upper-triangular matrices*, J. Algebra, 2004, vol. 278, 739-765.
- [12] Asharafi A.R., Koorepazan-Moftakhar F., *Towards the classification of finite simple groups with exactly three or four supercharacter theories*, arXiv:1605.08971
- [13] Benedett C., *Combinatorial Hopf algebra of supercharacters of tipe D* , Journal of algebraic Combinatorics, 2013, vol. 38, no. 4, 767-783.
- [14] Brumbaugh J.L., Bulkow M., Fleming P.S., Garcia L.A., Garcia S.R., Karaali G., Michal M., Turner A.P., Suh H. *Supercharacters, exponential sums and the uncertainty principle*, Journal of Number theory, 2014, vol. 144, 151-175.
- [15] Burkett Sh., Lamar J., Lewis M.L., Wynn C., *Groups with exactly two supercharacter theories*, arXiv:1506.00015
- [16] Diaconis P., Isaacs I.M., *Supercharacters and superclasses for algebra groups*, Trans.Amer.Math.Soc., 2008, vol. 360, 2359-2392.
- [17] Fowler C.F., Garcia S.R., Karaali G., *Ramanujan sums as supercharacters*, The Ramanujan Journal, 2014, vol.32, 205-241.
- [18] Grinberg D., Reiner V., *Hopf algebras in combinatorics*, preprint arXiv:1409.8356
- [19] Hendrickson A.O.F., *Supercharacter theory constructions corresponding to Schur ring products*, Comm. Algebra, 2012, vol. 40, no.12, 4420-4438.
- [20] Isaacs I.M., Karagueuzian D., *Involution and characters of upper triangular matrix groups*, Math. of Computation, 2005, 74:2027-2033.
- [21] Kazhdan D., *Proof of Springer hypothesis*, Israil J.Math., 1977, 28:272-284.
- [22] Kirillov A.A., *Lectures on the orbit method*, Graduate Studies in Math., vol. 64, 2004.
- [23] Kirillov A.A., *Variations on the Triangular Theme*, Amer. Math. Soc. Transl., 1995, 169:43-73
- [24] Curtis C.W., Reiner I., *Representation theory of finite groups and associative algebras*, 1962, New York, London: Interscience publishers.

- [25] Marberg E., Theim N., *Superinduction for pattern groups*, J. Algebra, 2009, vol. 321, 3681-3703.
- [26] Marberg E., *Combinatorial methods of character enumeration for the unitriangular group*, J. Algebra, 2011, vol. 345, 295-323
- [27] Otto B., *Constituents Of supercharacters and Kirillov functions*, Archiv der Mathematik, 2010, vol.94, 319-326.
- [28] Panov A.N. *Supercharacter theory for groups of invertible elements of reduced algebras*, 2016, St. Petersburg Math. J. 2016, vol. 27, 1035-1047.
- [29] Panov A.N., *Supercharacters for the finite groups of triangular type*, arXiv:1508.05767
- [30] Panov A.N., *Restriction and induction for supercharacters of finite groups of triangular type*, arXiv:1610.04846
- [31] Panov A.N., *Invariants of the coadjoint action on the basic varieties of the unitriangular group*, Transformation groups, vol.20, no. 1, 2015, 229-246.
- [32] Pierce R.S., Associative algebras, Springer-Verlag, New York, 1982.
- [33] Serr J.-P., Linear representations of finite groups, 1977, New-York: Springer-Verlag.
- [34] Thiem N., *Branching rules in the sing of super functions of unipotent upper-triangular matrices*, Journal of Algebraic combinatorics, 2010, vol. 31, 267-298.
- [35] Thiem N., Venkateswaran V., *Restricting supercharacters of the finite unipotent uppertriangular matrices*, Electronic Journal of Combinatorics, 2009, vol. 16, paper number 23
- [36] Zelevinsky A., Representations of classical groups. A Hopf algebra approach, Lecture Notes in Math. no. 869, Springer-Verlag, Berlin-Heidelberg-New York, 1981
- [37] Ning Yan, Representation Theory of finite unipotent linear groups, Ph.D. Thesis, Department of mathematics, University of Pennsylvania, 2001 (see also arXiv: 1004.2674).